STATISTICS MASTER'S/PH.D. QUALIFYING EXAM
January 10, 2000

DIRECTIONS:

Answer all questions. SHOW ALL WORK. GIVE REASONS FOR STEPS.

Tables are attached, along with a summary of the standard discrete and continuous distributions.

1. Let $X_1, X_2, \cdots, X_n$ be a random sample from a population with probability density function:

$$f(x|\theta) = \begin{cases} 2\theta x e^{-\theta x^2} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $\theta > 0$. You may need to use: $E(X^2) = 1/\theta$ and $Var(X^2) = 1/\theta^2$.

a. (10 points) We know from the exponential form of the density that $T = \sum_{i=1}^{n} X_i^2$ is a minimal, sufficient statistic. Demonstrate that T is (i) sufficient and (ii) minimal in this case, without using the general result for the one-parameter exponential family. State clearly the criteria you use.

b. (5 points) What does the information inequality (Cramér-Rao lower bound) tell you about the variance of any unbiased estimator of θ. (State the special case of the inequality for the given density).

c. (5 points) Define consistency and show that the statistic T (defined in part (a)) is a consistent estimator of $1/\theta$. State fully any theorem you use.

2. If $\theta > 0$ and you have a random sample of size n from a population with pdf:

$$f(x|\theta) = \begin{cases} 2\theta^2/x^3 & x > \theta \\ 0 & \text{elsewhere} \end{cases}$$

a. (5 points) Sketch the likelihood function.

b. (10 points) Find the maximum likelihood estimator of θ.

c. (5 points) Determine whether or not the estimator in part (b) is unbiased.

3. a. (5 points) What is meant by the most powerful test of size 0.05, of a simple null hypothesis H_0 against a simple alternative H_1?

b. (10 points) Derive an approximate rejection region for the most powerful test of size 0.05 of the null hypothesis H_0: p=.5 against the alternative H_1: p=.6, for a random sample of 100 Bernoulli random variables X_i (P(X_i=1)=p, P(X_i=0)=1-p, $i = 1, \cdots, 100$).

c. (5 points) For what set of alternatives is the test in (b) uniformly most powerful?

d. (5 points) Sketch the power function $\pi(p)$ of the test. Specify values for $\pi(.5)$ and (approximately) $\pi(.6)$.

4. (5 points) Suppose that 6% of the men and 4% of the women working for a corporation make over \$ 100,000 a year. 40% of the employees of the corporation are women. What percent of those who make over \$ 100,000 a year are men?

5. (10 points) The lifetime of a bulb is an exponential random variable with expectation equal to 3 years. What is the probability that in a sample of 1000 such bulbs there are at least 300 bulbs whose lifetime is more than 3 years?

6. (5 points) On average, the number of vacant rooms in a large hotel on a random day is 3. What is the probability that tomorrow there will be at least one vacant room? Try to explain your assumptions about the distribution of the number of vacant rooms.

7. (5 points) Let Z be a standard normal random variable. Find $E(Z^5)$, $E[\sin(Z)]$, $E\frac{Z}{1+Z^2}$.

8. a. (5 points) n missiles are fired at a target and hit it independently with probabilities $p_1, p_2, \ldots, p_n$. What is the probability that the target is hit?

b. (10 points) Let X be the number of missiles that hit the target. What is $E(X)$ and $\mathrm{Var}(X)$?