

STATISTICS MASTER'S/PH.D. QUALIFYING EXAM
August 10, 2000

DIRECTIONS: Answer all questions. Show all work. Give reasons for all steps and all answers. *This is an in class exam. You are to work it independently without consultation with any other being in any form. If you are confused about the wording of any question, contact the proctor.*

Tables are attached, along with a summary of the standard discrete and continuous distributions.

1. Suppose that you mixed thoroughly 100 raisins in dough and then baked out of it 25 rolls. Explain why in this case the number of raisins in a roll is approximately a Poisson random variable with parameter 4.
2. Suppose that among 100 fuses there are X defective ones, where X is a random variable that takes one of the values 0, 1, 2 with probability $1/3$ each. You pick at random 10 fuses out of 100 and find out that the number Y of defectives in the sample.
 - (a) Find $\Pr(Y = 0|X = 1)$.
 - (b) Find $\Pr(Y = 0|X = 2)$.
 - (c) Find $\Pr(Y = 0)$.
 - (d) If the number of defectives in the sample is 0, what is the probability that there are no defective fuses at all?
3. Let X be an exponential random variable with parameter $\theta > 0$ (i.e. $\mathbf{E}X = \frac{1}{\theta}$).
 - (a) Show that the moment generating function of X is equal to

$$\left(1 - \frac{t}{\theta}\right)^{-1}, \quad t < \theta.$$

- (b) Use this to compute $\mathbf{E}X^k$ for $k = 0, 1, 2, \dots$.
4. Suppose X_1, \dots, X_n is an iid sample from $f(x|\theta)$ given by

$$f(x) = \theta x^{\theta-1} \quad \text{for } 0 \leq x \leq 1, \quad \theta > 0$$

- (a) Find a sufficient statistic for θ . Justify your answer.
 - (b) Find the MLE for θ .
 - (c) Find the method of moments estimator for θ .
 - (d) Is your sufficient statistic minimal sufficient and/or complete?
 - (e) Is there a function of θ , say $g(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramer-Rao Lower Bound? If so, find it. If not, show why not.

5. Suppose $X \sim \text{Gamma}(n, 1)$ and $Y \sim \text{Gamma}(m, 1)$. Define

$$U = \frac{X}{X + Y} \quad V = X + Y$$

Find the joint density of (U, V) and use it to find the marginal density of U and the marginal density of V . Identify these distributions by name.

6. Suppose that we have two independent random samples: $X_1, \dots, X_n \sim \text{i.i.d. Exp}(\theta)$ (with mean θ) and $Y_1, \dots, Y_m \sim \text{i.i.d. Exp}(\mu)$ (with mean μ). (**This parameterization is different from Problem 3!**)

- (a) Identify the distribution of $\sum X_i$, $i = 1, \dots, n$ and $\sum Y_j$, $j = 1, \dots, m$. Give the name along with parameters.
- (b) Find the (generalized) LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.
- (c) Show that the test in (b) can be based on the statistic

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_j}.$$

- (d) Find the distribution of T when H_0 is true. (You may reference previous problems for details but you must name the distribution and have correct parameters for full credit.)
7. In the scientific literature, one often sees the following method used to test the hypothesis that two populations means, say μ_1 and μ_2 , are equal. First, compute individual 95% confidence intervals (CI) for μ_1 and μ_2 . If the two individual confidence intervals do not overlap, then reject the null hypothesis that $\mu_1 = \mu_2$. The researchers assume that this test has size $\alpha = .05$. Assume two independent normal random samples, each of size n and with a *known* common population variance σ^2 . The individual 95% CI's have the form $\bar{X}_i \pm 1.96\sigma/\sqrt{n}$, where \bar{X}_i is the sample mean for sample i ($i = 1, 2$).
- Show whether this test has the desired size (i.e. .05), or is it conservative (size $< .05$), or liberal (size $> .05$)?

HINT: Draw a picture