STATISTICS MASTER'S/PH.D. QUALIFYING EXAM, IN CLASS PORTION

January 7, 2002

This is a closed book, closed notes exam. There is a table of distributions accompanying this exam which will be handed out to you.

1. Let X and Y be continuous random variables such that X has pdf

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & otherwise \end{cases}$$

   Given $X = x$, Y is uniformly distributed on the closed interval $[0, x]$.

   (a) Find $P(XY < 0.2)$.

   (b) Find $E[X|Y = y]$.

2. Let $X_1, X_2, \ldots.X_n$ be a random sample from the continuous probability distribution with

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & 0 \le x < \infty \\ 0, & otherwise \end{cases}$$

   where $\theta \in (0, \infty)$.

   (a) Find the form of a complete, minimal sufficient statistic for θ.

   (b) Let $c > 0$ be an arbitrary positive constant and set $\tau_c(\theta) = P_\theta(X_1 \le c)$. Derive the UMVUE for $\tau_c(\theta)$.

   (c) Consider the estimator $\tilde{\tau}_c = 1 - e^{-nc/T}$ for $\tau_c(\theta)$, where $T = \sum_{i=1}^n X_i$. Argue (without actually finding the expected value of $\tilde{\tau}_c$) that $\tilde{\tau}_c$ is not unbiased for $\tau_c(\theta)$.

3. Suppose that Y is a log-normal random variable. That is, $Y = \exp(X)$ where $X \sim N(\mu, \sigma^2)$.

   (a) Show that $E(Y) = M_X(1)$ where $M_X(t)$ is the moment generating function of X.

   (b) Evaluate $E(Y)$ and $Var(Y)$.

   (c) Consider a non-negative random variable Z where $P(Z = 0) = p$ $(0 < p < 1)$ and the distribution of Z given that $Z > 0$ is log-normal. Find the cumulative distribution function of Z and evaluate $E(Z)$.

4. Suppose X and Y are independent, with $X \sim N(\mu_X, \sigma^2)$ and $Y \sim N(\mu_Y, \sigma^2)$, where σ^2 is known and $\mu_Y \neq 0$. Let $\theta = \mu_X / \mu_Y$.

   (a) Given independent samples $X_1, \ldots, X_n$ from X and $Y_1, \ldots, Y_m$ from Y, show that

$$\frac{\bar{X} - \theta \bar{Y}}{\sigma \sqrt{\frac{1}{n} + \theta^2 \frac{1}{m}}} \sim N(0, 1).$$

   (b) Using the result for (a), develop a procedure for computing a 95% confidence region for θ. Be explicit about the form of the region; for example, under what conditions is the region an interval?

5. Suppose that $X_1, \ldots, X_n$ are independent Bernoulli(θ) random variables (i.e. $P(X_i = 0) = 1 - \theta$ and $P(X_i = 1) = \theta$), where θ is either 1/3 or 2/3. Assume n is odd.

   (a) Find the maximum likelihood estimator (MLE) of θ.

   (b) Show directly that the MLE is consistent for θ.

   (c) Find the form of the most powerful test of $H_0 : \theta = 1/3$ against $H_A : \theta = 2/3$.

6. Consider the simple linear regression problem with an intercept of zero:

$$Y_i = \beta X_i + \epsilon_i, \qquad \epsilon_1, \ldots, \epsilon_n \overset{iid}{\sim} N(0, \sigma^2).$$

   Here, the $X_1, \ldots, X_n$ are known, fixed covariates.

   (a) Derive the maximum likelihood estimator (MLE) of β, $\hat{\beta} = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2}$.

   (b) Show directly that $E(\hat{\beta}) = \beta$.

   (c) Show directly that $Var(\hat{\beta}) = \frac{\sigma^2}{\sum_{i=1}^n X_i^2}$.

   (d) What is the distribution of $\hat{\beta}$?