STATISTICS Ph. D. COMPREHENSIVE EXAM August, 2002

General Instructions: Write your ID number on your answer sheets. Do not put your name on any of your answer sheets. All solutions should be rigorously explained. Show all work.

Reminder: All assertions should be rigorously proved.

1. Let $\{A_n\}$ be a sequence of independent events with $\mathbf{P}(A_n) = p_n$. Let ν denote the smallest number n such that A_n occurs (if none of the events occurs, $\nu = +\infty$). Find a necessary and sufficient condition for $\nu < +\infty$. Under this condition, find the probability mass function of the random variable ν.

2. Let X_λ be a Poisson random variable with parameter $\lambda > 0$. Show that
$$\frac{X_\lambda - \lambda}{\sqrt{\lambda}}$$
converges in distribution to a standard normal random variable as $\lambda \to \infty$.

3. Consider the general Gauss-Markov model
$$Y = X\beta + \epsilon,$$
where Y is an $n \times 1$ vector, X is an $n \times p$ matrix of rank p, β is a $p \times 1$ vector, and $\epsilon \sim N_n(0, \sigma^2 V)$, where V is a known symmetric positive definite matrix. The associated estimate of β and sum of squares residual are
$$\hat{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}Y \text{ and } SSR(Y) = (Y - X\hat{\beta})'V^{-1}(Y - X\hat{\beta}).$$
   (a) Show that $\hat{\beta} \sim N_p(\beta, (X'V^{-1}X)^{-1}\sigma^2)$.

   (b) Show that $\hat{\beta}$ and $SSR(Y)$ are independent if $X'V^{-1}Y$ and $V^{-1}(Y - X\hat{\beta})$ are independent. (Hint: consider inserting the identity matrix $I_n = VV^{-1}$ in the formula for $SSR(Y)$).

   (c) Show directly that $X'V^{-1}Y$ and $V^{-1}(Y - X\hat{\beta})$ are independent random vectors.

4. Formulate and prove the Information Inequality. Hint: This is the one that bounds from below variance of specific estimates.

5. Consider the classical problem of hypothesis testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ in the exponential family

$$f(x|\theta, \nu) = C(\theta, \nu) \exp[\theta U(x) + \nu T(x)].$$

Recall sufficient assumptions for existence of a UMP unbiased test, write down the critical function, draw a typical power function, and then prove that the test is UMP unbiased.

6. Suppose that $Y_i \sim \text{Poisson}(\mu_i)$ for $i = 1, 2, \ldots, k$ where

$$\log(\mu_i) = \beta_0 + \beta_1 x_i$$

for some $\beta = (\beta_0, \beta_1)'$ in $\mathbf{R^2}$. Here $x_1, x_2, \ldots, x_k$ are fixed covariate values.

(a) Derive the likelihood equations for obtaining the maximum likelihood estimate (MLE) $\hat{\beta}$ for β. Discuss how you would compute $\hat{\beta}$ for a given set of data.

(b) Derive the (expected) Fisher Information matrix for β.

(c) What is the large sample distribution of $\hat{\beta}$.

7. (Continuation of Problem 6).

(a) Suppose you wish to test $H_0 : \beta_1 = 0$ against the alternative $H_1 : \beta_1 > 0$. Give the form of an exact test of size (no greater than) α. Discuss some properties of this test.

(b) Suppose k is large. How would you approximate the critical value for the test in (a)? Be precise.

Remark: Consider the distribution of $Y (= Y_1, Y_2, \ldots, Y_k)'$ given $\sum_{i=1}^{k} Y_i$ under H_0.