Statistics Masters and Ph.D. Qualifying Exam
In Class: Wednesday August 13, 2003

Instructions: The exam has 5 multi-part problems of **equal** value. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete.

1. Suppose X and Y are independent $N(0, \sigma^2)$ random variables

   (a) Show that $X + Y$ and $X - Y$ are uncorrelated.

   (b) Show $Pr(X < Y) = 1/2$.

   (c) Find the probability density function of $W = \frac{X}{Y}$.

   (d) Are W and $V = \frac{Y}{X}$ identically distributed? Explain without deriving the distribution of V.

2. Suppose that (X, Y) is a bivariate random variable and define $\theta = Pr(X < Y)$.

   (a) Define the function $H(X, Y)$ to take the value 1 if $X < Y$ and 0 otherwise. Show that $E\{H(X, Y)\} = \theta$.

   (b) Assume that (X_i, Y_i) for $i = 1, 2, \ldots, n$ are independent with the same distribution as (X, Y). Set

   $$T = \sum_{i=1}^{n} H(X_i, Y_i).$$

   What is the (exact) distribution of T? Be explicit.

   (c) In n is large, what does the Central Limit Theorem say about the approximate distribution of T/n?

   (d) Using part (c), how would you construct an approximate 95% confidence interval for θ?

3. Suppose $Y_1, Y_2, \ldots, Y_n$ are independent with $Y_i \sim N(\beta X_i, 1)$ for $i = 1, 2, \ldots, n$. Here β is an unknown parameter and $X_i > 0$ is a known constant.

   (a) Use the factorization theorem to obtain a sufficient statistic for β.

   (b) Compute the maximum likelihood estimate (MLE) of β, say $\hat{\beta}$.

   (c) Show that $\hat{\beta}$ is unbiased for β and find $Var(\hat{\beta})$.

   (d) Another unbiased estimator of β is $\tilde{\beta} = \frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} X_i}$. Compute $Var(\tilde{\beta})$.

   (e) Which estimator $\hat{\beta}$ or $\tilde{\beta}$ is preferred? Explain.

4. The probability mass function (pmf) for a multinomial random vector (X_1, X_2, X_3) with parameters n and $\mathbf{p} = (p_1, p_2, p_3)$ is given by

$$p(i, j, k) = Pr(X_1 = i, X_2 = j, X_3 = k) = \frac{n!}{i!j!k!} p_1^i p_2^j p_3^k,$$

when $i+j+k = n$, $p_1+p_2+p_3 = 1$, $0 \leq i, j, k \leq n$ and $0 \leq p_1, p_2, p_3 \leq 1$; the pmf is zero otherwise. Alternatively, since knowing X_1 and X_2 implies you know X_3, we may write

$$p(i, j) = Pr(X_1 = i, X_2 = j) = \frac{n!}{i!j!(n-i-j)!} p_1^i p_2^j (1 - p_1 - p_2)^{(n-i-j)},$$

with the obvious restrictions on i, j, p_1, and p_2.

   (a) Let $Y_1 \sim \text{Binomial}(n, p_1)$ and $Y_2 | Y_1 \sim \text{Binomial}(n - Y_1, p_2/(p_2 + p_3))$. Show that (Y_1, Y_2) and (X_1, X_2) have the same distribution by explicitly calculating the joint distribution of (Y_1, Y_2).

   (b) If you need to simulate from a multinomial distribution with parameters n and $\mathbf{p} = (p_1, p_2, p_3)$, but only have a way to simulate from the binomial distribution, how could you accomplish this?

Table 1: Probability Distribution for Problem 5.

x :	0	1	2	3	4	5
$f(x;\theta=0)$	.05	.05	.10	.10	.20	.50
$f(x;\theta=1)$	.10	.15	.25	.15	.25	.10

5. Suppose X is a random variable with probability mass function $f(x;\theta)$, where θ is either 0 or 1. The probability distribution for X is given in Table 1 above.

   (a) Find the most powerful test of size .10 for testing $\theta = 0$ against $\theta = 1$.

   (b) Compute the power for the test found in (a).