STATISTICS PH.D. COMPREHENSIVE EXAM

January 12, 2004 2-5 PM

Directions: The exam consists of eight questions of equal point value. You are required to attempt problems 1 through 4 and three of the remaining 4 problems (5, 6, 7, and 8). Make sure to write your ID number (Social Security number) on your answer sheets. Do not put your name on any of your answer sheets. All solutions should be complete and rigorous. If you cannot solve a problem, try to outline an approach that may lead to a solution.

1. Consider a linear model

$$Y = X\beta + e, \quad E(e) = 0, \quad Cov(e) = \sigma^2 I.$$

State and prove the Gauss-Markov Theorem.

2. Consider a linear model

$$Y = X\beta + e, \quad E(e) = 0, \quad Cov(e) = \sigma^2 V,$$

where V is positive definite.

(a) If X is $n \times p$ with $r(X) = r$, what is the rank of the null space of X?

(b) Show that the null space of X equals the null space of VX.

(c) What is $r(VX)$?

(d) If $C(VX) \subset C(X)$, show that $C(VX) = C(X)$.

(e) Show that $C(V^{-1}X) = C(X)$.

(f) Define the oblique projection operator onto $C(X)$, $A = X(X'V^{-1}X)^{-}X'V^{-1}$. Show that for any vector w, $w \perp C(X)$ implies $Aw = 0$.

(g) Show that A is the perpendicular projections operator onto $C(X)$.

(h) Show that if $C(VX) = C(X)$, then least squares estimates are BLUES for this model.

3. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with $EX_j = 0$ and $EX_j^4 < +\infty$. Prove that for some constant $C > 0$

$$E|X_1 + X_2 + \ldots + X_n|^4 \leq Cn^2.$$

Use it to show that

$$\frac{X_1 + \ldots + X_n}{n} \to 0$$

as $n \to \infty$ with probability 1.

4. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density f. Let

$$S_n := X_1 + \ldots + X_n.$$

(a) What is the density of S_2? Of S_n?

(b) For $1 \leq j \leq n$, find the conditional density of X_j given S_n.

(c) Compute $E(X_1|S_n)$.

5. Consider the problem of comparing variances of two normal distributions. Present necessary theoretical background (such as unbiasedness, similarity, Neyman structure, completeness, etc. if necessary), to develop classical hypothesis tests for this setting.

6. Can the theory developed in problem 5 be applied to unbiased confidence interval estimation? Give definitions and an example.

7. State and prove Basu's Theorem. (Give definitions for all necessary statistical notion.)

8. Estimation based on the score function is fundamental in statistics. Present conditions, all necessary definitions and a proof of the following result. Consider n iid realizations from a distribution P_θ and assume that a $\sqrt{n}$-consistent estimator $\tilde{\theta}_n$ of θ exists. Then the estimator sequence

$$\delta_n = \tilde{\theta}_n - \frac{L'(\tilde{\theta}_n)}{L''(\tilde{\theta}_n)}$$

is asymptotically efficient. Here $L(\theta)$ is the log–likelihood.