Statistics Masters and Ph.D. Qualifying Exam
In Class: Monday January 12, 2004

Instructions: The exam has 5 multi-part problems, each worth 20 points. All of the problems will be graded. Write your ID number (Social Security Number) on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. A STANDARD NORMAL TABLE IS PROVIDED AT THE END OF THE EXAM.

1. To play in the multi-state **Powerball** Lottery, a contestant purchases a 1 dollar ticket and chooses 5 distinct numbers from the integers 1 through 49. The 6^{th} number, or Powerball, is then chosen from a separate list of the integers 1 through 42. To win the jackpot, which was 210 million dollars for the December 31, 2003 drawing, you must match each of the first five numbers (the order of this sequence is unimportant), and match the Powerball. A player can select any sequence of admissible numbers, so several individuals could potentially match the winning set of numbers. In such cases the jackpot is split among the individuals holding a winning ticket.

   (a) What is the chance of winning (or sharing) the jackpot, if you purchase one ticket and choose the 6 numbers at random?

   (b) Suppose you purchase 100 tickets, and choose the 6 numbers on each ticket at random, and independently from one ticket to the next. Give an expression for the probability that you win (or share) the jackpot.

   (c) A smaller prize is given if you pick 4 of the first 5 numbers correctly. What is the chance of winning this prize, if you purchase one ticket and select the 6 numbers at random?

2. Suppose that the joint density of random variables X and Y is given by

$$f(x,y) = \frac{xy}{96}, \text{ for } 0 < x < 4, 1 < y < 5.$$

(a) Compute the mean and variance of $U = X + 2Y$.

(b) Find the density of U. (*Hint: use the Jacobian method of transformation and pay very careful attention to limits*)

3. Suppose that $X_1, ..., X_n$ $(n > 3)$ are independent observations from a Bernoulli(θ) distribution, that is $P[X_i = 1] = 1 - P[X_i = 0] = \theta$.

   (a) Show that $\bar{X} = \sum_{i=1}^{n} X_i/n$ is a sufficient estimator (statistic) of the parameter θ.

   (b) Show that the estimator Y is not a sufficient estimator of θ where
   $$Y = \frac{X_1 + 2X_2 + 3X_3}{6}.$$

   (c) Considering issues such as bias, variance, consistency, and mean square error, which of the estimators $\bar{X}$ and Y is to be preferred? Explain in detail.

4. (a) State the Neyman-Pearson Lemma.

   (b) Suppose that $X_1, X_2, \ldots, X_{25}$ are a random sample of size 25 from a normal distribution with mean θ and variance 100. Use the Neyman-Pearson lemma to find the most powerful test of size $\alpha = 0.05$ for testing $H_0 : \theta = 75$ against $H_1 : \theta = 80$.

   (c) Compute the power of the most powerful test of size $\alpha = 0.05$ for testing $H_0 : \theta = 75$ against $H_1 : \theta = 80$.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $U(0, \theta)$ distribution, i.e. $f(x|\theta) = \frac{1}{\theta} I_{(0,\theta)}(x)$. Given this sample,

   (a) Does this family of pdfs belong to the exponential family?

   (b) What is a MLE estimator for θ? Justify your answer.

   (c) Is your estimator from part (b) an unbiased estimator for θ? Justify your answer.

   (d) Show explicitly that the MLE of θ converges in probability to θ as $n \to \infty$.