Statistics Masters and Ph.D. Qualifying Exam
In Class: Monday August 9, 2004

Instructions: The exam has 6 multi-part problems. The point value for each problem is highlighted. All of the problems will be graded. Write your ID number (last 4 digits of your Social Security Number) on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete.

1. (15 points) A bowl contains twenty cherries, exactly fifteen of which have had their stones (pits) removed. A greedy pig eats five whole cherries, picked at random, without remarking on the presence or absence of stones. Subsequently, a cherry is picked randomly from the remaining fifteen.

   (a) What is the probability that this cherry contains a stone?

   (b) Given that this cherry contains a stone, what is the probability that the pig consumed at least one stone?

2. (20 points) Suppose that X and Y have a joint distribution

$$f(x,y) = cx^2y \quad \text{for} \quad 0 < x^2 \leq y \leq 1.$$

   (a) Determine the value of c that makes this a density.

   (b) Find $P(X \geq Y)$. (Big hint: draw a picture before you do anything).

   (c) Find the marginal density of X. Pay attention to limits.

   (d) Find the conditional density of Y given that $X = x$. Pay attention to limits.

   (e) Find $P(Y \geq \frac{3}{4} | X = \frac{1}{2})$.

3. (10 points) Suppose that random variable X has a MGF given by

$$M(s) = \frac{1}{5}e^{s} + \frac{2}{5}e^{4s} + \frac{2}{5}e^{8s}.$$

   Find the probability distribution of X.

4. (15 points) Let $X \sim N(\mu, \sigma^2)$ and let $Y \sim N(\gamma, \sigma^2)$. Suppose X and Y are independent. Define $U = X + Y$ and $V = X - Y$.

   (a) Find the mean and variance of U and V, respectively.

   (b) Show that U and V are independent random variables.

   (c) Find the marginal distribution of each of them.

5. (20 points) Suppose $X_1, ..., X_n$ $(n > 1)$ are i.i.d. Poisson(λ) random variables where the Poisson probability mass function is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2..., \text{ and } \ 0 \le \lambda < \infty.$$

(a) Find the distribution of $\sum_{i=1}^n X_i$.

(b) Find a sufficient statistic for λ and justify your choice.

(c) Is the family of distributions of $\sum_{i=1}^n X_i$ complete? Justify your answer.

(d) Find a method of moments (MOM) estimator for λ. Is this estimator unique? If so explain why it is unique. If not, find another MOM estimator for λ.

(e) Find the score function $S(\lambda)$.

(f) Find the maximum likelihood estimator of λ.

6. (20 points) Continuation of last problem.

(a) Find the information in the sample, $I(\lambda)$.

(b) Find the Cramer-Rao Lower Bound on the variance of all unbiased estimators of λ.

(c) Prove that $\bar{X}$ is the best unbiased estimator of λ without using the Cramer-Rao theorem.

(d) Prove that the sample variance $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$ is also an unbiased estimator of λ.

(e) Prove that $E(S^2|\bar{X}) = \bar{X}$.

(f) Which estimator of λ, $\bar{X}$ or S^2, do you prefer? Justify your answer.