Statistics Masters and Ph.D. Qualifying Exam
In Class: January 10, 2005

Instructions: The exam has 6 multi-part problems of **equal** value. All of the problems will be graded. Make sure to write your ID number (last 4 digits of your Social Security number) on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete.

1. Let X and Y have a joint density function given by

$$f(x, y) = 3x, \ 0 \leq y \leq x \leq 1$$

and $f(x, y) = 0$ otherwise.

(a) Find the marginal density functions of X and Y.

(b) Find $P(X \leq .75 | Y \leq .50)$.

(c) Find the conditional density of X given $Y = y$.

(d) Find $E(Y|X = x)$.

2. Suppose the distribution of Y, conditional on $X = x$, is $N(x, x^2)$ and that the marginal distribution of X is uniform(0,1).

(a) Find the expected value of Y, $E(Y)$.

(b) Find the variance of Y, $V(Y)$.

(c) Find the covariance between X and Y, $Cov(X, Y)$.

(d) Prove that Y/X and X are independent.

3. Suppose $X_i, \ i = 1, 2$ are independent gamma(α_i, β) random variables, with density functions

$$f(y|\alpha_i, \beta) = \frac{1}{\Gamma(\alpha_i)\beta^{\alpha_i}} y^{\alpha_i - 1} exp(-y/\beta) \ \text{ for } \ y > 0;$$

and 0 otherwise. Let

$$Y_1 = X_1 + X_2 \quad \text{and} \quad Y_2 = \frac{X_1}{X_1 + X_2}.$$

(a) Find the joint distribution of Y_1 and Y_2.

(b) Find the marginal distribution of Y_1 and the marginal distribution of Y_2.

4. Suppose $X_1, X_2, \ldots, X_n$ are iid $N(\mu, \sigma^2)$.

   (a) Find a minimal and sufficient statistic for the pair (μ, σ^2).

   (b) Assume that μ is completely known. Find an expression and make a sketch of the likelihood function for σ^2.

   (c) Under the same assumption as (b), what is the MLE of σ^2?

   (d) Find the variance of the MLE estimator found in (c). Is this a consistent estimator of σ^2?

   (e) Find a $(1-\alpha) \times 100\%$ confidence interval for σ^2 for the cases of μ known and μ unknown. Can you make any distinctions between these intervals?

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from the geometric density

$$f(x|\theta) = \theta(1-\theta)^x; \quad x = 0, 1, 2, \ldots$$

   where $0 < \theta < 1$.

   (a) Does this pdf belong to the exponential family?

   (b) Find a method of moments estimator for θ.

   (c) Find the Cramer-Rao lower bound for the variance of unbiased estimators of $(1-\theta)$.

   (d) Find the UMVUE of $(1-\theta)/\theta$ if such exists.

6. Let X be one observation of the exponential density

$$f(x|\theta) = \theta \exp(-\theta x); \quad x > 0; \quad \theta > 0$$

   and assume we wish to test, $H_0 : \theta_0 = 0.5$ versus $H_1 : \theta_1 = 1$.

   (a) Find the acceptance region for H_0 of the level α likelihood test.

   (b) Obtain the p-value associated with $X = 3$.

(c) Suppose that apriori $p(H_0) = p(H_1) = 0.5$. For $X = 3$ calculate the posterior probability of H_0: $p(\theta_0 = 0.5|X = 3)$. Do you reject or accept the null hypothesis?

(d) Compare the results of the test in item (a) with those from the test in item (c).