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Statistics Masters and Ph.D. Qualifying Exam
In Class: Tuesday August 16, 2005

Instructions: The exam has 5 multi-part problems of **equal** value. All of the problems will be graded. Write your ID number on your answer sheets (last 4 digits of your SSN). Do not put your name on any of the sheets. Be clear, concise, and complete.

1. Let X be the number obtained from a single roll of a fair six-sided die. Given the value of $X = x$, roll a second fair die with x sides, numbered 1,2, ... x. Let Y denote the number obtained on the roll of the second die.

   (a) Find the joint probability function for X and Y.

   (b) Are the random variables X and Y independent? Justify your answer.

   (c) Evaluate $P(Y > X - 2)$.

   (d) Compute $E[Y|X = x]$ and $Var[Y|X = x]$.

   (e) Use (d) to evaluate $E[Y]$.

2. Consider a sequence of days, and let R_i denote the event that it rains on day i, $i = 0, 1, \ldots$.. Similarly, let N_i denote the event "no rain" on day i. Assume that $P(R_i|R_{i-1}) = \alpha$ and $P(N_i|N_{i-1}) = \beta$ for all $i > 0$ and that day 0 is today. Suppose further that only day $i-1$'s weather is relevant to predicting day i's; that is, expressions like $P(R_i|R_{i-1} \cap R_{i-2} \cap \ldots \cap R_0)$ are equivalent to $P(R_i|R_{i-1})$.

   (a) If the probability of rain today (day 0) is p, what is the probability of rain tomorrow?

   (b) What is the probability of rain the day after tomorrow?

   (c) What is the probability of rain n days from now?

   (d) (Extra Credit) What happens in (c) as n approaches infinity?

3. Let X_1 and X_2 be independent and identically distributed random variables with a $U(0, 1)$ distribution.

   (a) Find the cumulative distribution function (cdf) of $Z = X_1 X_2$. Hint: You might first consider finding the distribution function of $\log(Z)$.

   (b) Use part (a) to find the density function of Z.

   (c) Let $Y = X_1 + X_2$. Find the conditional probability that $Y \leq \frac{1}{2}$, given that $Z \leq \frac{1}{16}$.

4. Let $X_1, \ldots, X_n \overset{iid}{\sim} f_X(x|\theta) = \dfrac{x^2}{\theta^3}\sqrt{\dfrac{2}{\pi}} \exp\left(-\dfrac{x^2}{2\theta^2}\right)$ for $x \geq 0$ and $\theta > 0$. This is called the *Maxwell distribution.* Note that $E(X_1) = \theta\sqrt{\dfrac{8}{\pi}}$ and $E(X_1^2) = 3\theta^2$.

   (a) Find a complete, sufficient statistic for θ.

   (b) What is the expectation of the statistic found in part (a)?

   (c) Find an uniformly minimum variance unbiased estimator (UMVUE) for θ^2.

(d) Find the maximum likelihood estimator (MLE) for θ; call it $\hat{\theta}$. Find the MLE for θ^2.

(e) Find the Cramer-Rao lower bound (CRLB) for unbiased estimators of θ^2.

(f) Does the estimator in part (c) achieve the CRLB for estimating θ^2?

5. Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, 1)$.

   (a) Find a uniformly most powerful (UMP) test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ for $\theta_0 < \theta_1$. State the rejection region R in terms of the significance level α and a sufficient statistic for θ.

   (b) Find the likelihood ratio test (LRT) statistic $\lambda(\mathbf{x})$ for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

   (c) What is the exact distribution of $-2 \log \lambda(\mathbf{X})$ when H_0 is true?

   (d) Say that $-2 \log \lambda(\mathbf{x}) = 14.3$ is observed. Do you reject or not reject H_0 at the $\alpha = 0.05$ level? Note: $P(\chi_1^2 < 3.84) = 0.95$.