

STATISTICS Ph. D. COMPREHENSIVE EXAM
January 2006

Write your ID number on your answer sheets. Do not write your name. Give complete solutions of the problems with all necessary definitions, proofs and explanations.

STATISTICAL INFERENCE

1. State and prove the Rao-Blackwell Theorem.
2. State and prove the Gauss-Markov Theorem.
3. Consider a linear model $Y = X\beta + e$ with $e \sim N(0, \sigma^2 I)$ and let M be the perpendicular projection operator (ppo) onto $C(X)$.
 - (a) Find the distribution of $\hat{e} \equiv (I - M)Y$.
 - (b) Show that \hat{e} and $\tilde{Y} \equiv MY$ are independent.
 - (c) Find the distribution of Y conditional on \tilde{Y} . [Hint: Write $Y = \tilde{Y} + \hat{e}$.]

While still assuming that $Y = X\beta + e$ is the correct model, consider fitting a model

$$Y = X\beta + Z\gamma + e$$

where Z is an $n \times r$ matrix. Let P be the ppo onto $C(X, Z)$.

- (d) In terms of quadratic forms and projection operators, give the numerator and denominator sums of squares for the F test of $H_0 : \gamma = 0$.

While still assuming that $Y = X\beta + e$ is the correct model, consider fitting a model

$$Y = X\beta + \hat{Z}\gamma + e$$

where \hat{Z} is an $n \times r$ matrix which is a function of \tilde{Y} . Consider testing $H_0: \gamma = 0$ with the test statistic of part (d) using \hat{Z} to replace Z .

- (e) Find the distributions of the of the numerator and denominator sums of squares given \tilde{Y} .
 - (f) Show that the numerator and denominator sums of squares are independent given \tilde{Y} .
 - (g) Find the distribution of the F statistic given \tilde{Y} .
 - (h) Find the unconditional distribution of the F statistic. [This is trivial if you think about it the correct way.]
4. Consider a sequence of iid random vectors $(y_{i1}, y_{i2})'$ with means $(\mu_1, \mu_2)'$ and covariance matrix parameters $\sigma_{11}, \sigma_{22}, \sigma_{12}$.
- (a) Given an estimate of $(\mu_1, \mu_2)'$ and find its asymptotic distribution.
 - (b) Give an estimate of μ_1/μ_2 and use the Delta method to find its asymptotic standard distribution.
 - (c) Show that the usual t statistic on y_1 , i.e., $(\bar{y}_{\cdot 1} - \mu_1)/(s/\sqrt{n})$ converges in law to a standard normal.
5. Give examples of the following
- (a) A sequence of random variables that converge in probability but do not converge almost surely.
 - (b) A sequence of random variables that converge in probability but do not converge in mean square.
6. Suppose X_n is distributed uniformly on the numbers $j/n, j = 1, \dots, n$. Show that X_n converges in law to a $U(0, 1)$ distribution.
7. Show that if $\sum E(X_n - X)^r < \infty$ for $r \geq 2$, then X_n converges to X almost surely. Hint: Show that there is 0 probability of the sets A_n occurring infinitely often, where $A_n = \{|X_n - X| \geq \epsilon\}$.

8. Consider a decision problem with loss function $L(\theta, a)$ and data X conditional on θ .

- (a) Define the risk function and what it means to be a minimax rule.
- (b) Show that δ_0 is a minimax rule if for any θ_* and any rule δ ,
 $R(\theta_*, \delta_0) \leq \sup_{\theta} R(\theta, \delta)$
- (c) For loss function $L(\theta, a) = w(\theta)(\theta - a)^2$ with $w(\theta) > 0$, show that the Bayes decision rule is $E[w(\theta)\theta|X]$.

9. Consider a pdf (here C is a constant)

$$f(x_1, x_2, x_3, x_4) = C[19/18 + x_1 + x_2^2 + x_3^2 x_4^2 + x_1 \sin(2\pi x_4)] \prod_{l=1}^4 I(x_l \in [0, 1]).$$

Find:

- (a). $P(X_1 < 1/2, X_3 < 1/2, X_4 > 3/4)$.
- (b). $E\{X_1 X_4\}$.

10. Given that X and Y are independent RVs with the pdf $f_X(x)$ and $f_Y(y)$, respectively, find the pdf of $Z = X/Y$.

11. Consider a pdf $f_X(x) = Ce^{-|x|}I(-1 < x < 3)$. Set $Y = X^2$ and calculate:

- (a) the cdf of Y .
- (b) the pdf of Y .