Statistics Masters and Ph.D. Qualifying Exam
In Class: August 7, 2006

Instructions: The exam has 5 multi-part problems of **equal** value. All of the problems will be graded. Make sure to write your ID number (last 4 digits of your Social Security number) on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. Calculators are allowed. A NORMAL PROBABILITY TABLE IS PROVIDED.

1. Two litters of a particular rodent species have been born, one with two brown–haired and one gray–haired offspring (litter 1), and the other with three brown–haired and two gray–haired offspring (litter 2). We select a litter at random and then select an offspring at random from the selected litter.

(a) What is the probability that the animal chosen is brown–haired?

(b) Given that a brown–haired offspring was selected, what is the probability that the sampling was from litter 1?

(c) Suppose that the first animal chosen is removed from its litter, and the process of selecting a litter at random and then selecting an offspring at random from the selected litter is repeated. What is the probability that the second animal chosen is brown–haired (without any knowledge of the color of the first animal selected)?

2. An electronic device has a lifetime X (in 1000 unit hours) with probability density $f(x) = \exp(-x)$ for $x > 0$, and 0 otherwise. Suppose that a single device cost the manufacturer \$25 to make and is sold for \$30.

(a) The manufacturer provides a money back guarantee for any unit that fails within the first 100 hours of use. Assuming each unit that fails within the warranty period is returned for a refund, what is the manufacturer's expected profit, and the variance of the profit, per unit of sale? Give approximate dollar figures using the Taylor series approximation: $\exp(t) \approx 1 + t$, or give an exact value if you have a calculator.

(b) Suppose a store sells 100 units in a given month. Assuming that failures from unit to unit are independent, what is the distribution of the number of these 100 units that will fail during the warranty period? Be precise.

(c) Estimate (i.e. give a numerical value, not a formula) the probability that at least 15 of the 100 units will fail while under warranty.

3. Suppose X and Y are independent exponential random variables with densities given by $f(w) = \lambda^{-1}\exp(-w/\lambda)$ for $w > 0$, and zero otherwise, where $\lambda > 0$.

(a) Find the cumulative distribution function of $Z = X - Y$.

(b) Find the density function of Z.

(c) Find the moment generating function of Z.

4. Suppose X has density function

$$f(x;\theta) = 2\theta x + 2(1-\theta)(1-x)$$

for $0 < x < 1$ and 0 otherwise, where $0 \leq \theta \leq 1$ is an unknown parameter.

(a) Given a single observation from this distribution, write down a general form for the rejection region of a most powerful test of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_0 < \theta_1$.

(b) Suppose $\theta_0 = 0$ and $\theta_1 = 1$. Give an exact form for the critical region of the most powerful test with size $\alpha = .1$.

(c) Find the power of the test in (b).

(d) Is the test you found in (a) a most powerful test of its size for $H_0 : \theta = 0$ against $H_1 : \theta > 0$? Explain.

5. Let $X_1, \ldots, X_n \overset{iid}{\sim} f_X(x|\theta) = \dfrac{\theta}{x^{\theta+1}}$ for $x \geq 1$ and $\theta > 0$.

(a) Find a minimal sufficient and complete statistic for θ.

(b) What is $E(X_1)$? Does this expectation exist for all $\theta > 0$?

(c) Find the MOM estimator of θ, assuming $\theta > 1$.

(d) Find the MLE of θ.

(e) Find the MLE of $1/\theta$.

(f) Is either the MLE for θ or $1/\theta$ an unbiased estimator? Justify your answer.