Statistics Masters and Ph.D. Qualifying Exam
In Class: Monday August 6, 2007

Instructions: The exam has 5 multi-part problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.

1. Let X_1, X_2 be independent exponential(β) random variables (The exponential density as $f(x) = \frac{1}{\beta}e^{-(x/\beta)}$). Define $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = \sqrt{X_1 X_2}$.

   (a) What is the joint distribution of (Y_1, Y_2)?

   (b) What is the marginal distribution of Y_1? What is $\mathrm{E}(Y_1)$?

   (c) Find the conditional distribution of Y_2 given $Y_1 = y_1$. What is $\mathrm{E}(Y_2 \mid Y_1 = y_1)$?

   (d) What is $\mathrm{E}(Y_2)$?

2. Consider a random sample $X_1, X_2, \ldots, X_n$ from a population with mean μ and variance σ^2 and let $\bar{X}$ be the sample mean. Show that

   (a) $E(\bar{X}) = \mu$ and $Var(\bar{X}) = \sigma^2/n$

   (b) Show formally that if each $X_i \sim N(\mu, \sigma^2)$, then $\bar{X}$ also follows a Normal distribution.

   (c) What is the distribution of $\bar{X}$ if each X_i follows an exponential distribution with parameter β as given in problem 1?

   (d) In the context of part (c), use the moment generating function to show that the asymptotic distribution of $\bar{X}$ is Normal.

3. Suppose $Y_1, Y_2, \ldots, Y_n$ are independent with $Y_i \sim N(\beta X_i, \sigma^2)$ for $i = 1, 2, \ldots, n$. Here β is an unknown parameter, $X_i > 0$ is a known constant and $\sigma^2 > 0$ is known .

   (a) Find a sufficient statistic for β. Is this a minimal sufficient statistic?

   (b) Compute the maximum likelihood estimate (MLE) of β, say $\hat{\beta}$.

   (c) Show that $\hat{\beta}$ is an unbiased for β.

   (d) Show that another unbiased estimator of β is $\tilde{\beta} = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}$.

   (e) Which estimator $\hat{\beta}$ or $\tilde{\beta}$ is preferred? Justify your answer.

   (f) Find a $1 - \alpha$ confidence interval for β.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform(θ,$\theta + 1$) distribution. To test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$ use the test: reject H_0 if $Y_n \geq 1$ or $Y_1 \geq k$, where k is a constant, $Y_1 = \min\{X_1, \ldots, X_n\}$, $Y_n = \max\{X_1, \ldots, X_n\}$.

(a) Determine k so that the test will have size α.

(b) Find an expression for the power function of the test in part (a).

(c) Prove that the test is UMP size α.

(d) Describe how would you calculate a $1-\alpha$ confidence interval for θ. Give an expression for your interval.

5. Prove or Disprove the following statements for the events A, B, C. To disprove give a counterexample:

   (a) If $P(A) \leq P(B)$, then $P(A \mid C) \leq P(B \mid C)$.

   (b) If $P(A \cap B \mid C) = P(A \mid C)P(B \mid C)$, then $P(A \cap B \mid \bar{C}) = P(A \mid \bar{C})P(B \mid \bar{C})$.

   (c) If $P(A \cap B \mid C) = P(A \mid C)P(B \mid C)$, then $P(A \cap \bar{B} \mid C) = P(A \mid C)P(\bar{B} \mid C)$.

   (d) Suppose A and B are independent. If also A and B are conditionally independent given C, then A, B, C are mutually independent.