Statistics Masters and Ph.D. Qualifying Exam
In Class: Monday January 14, 2008

Instructions: The exam has 6 multi-part problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.

1. A box contains 5 coins and each has a different probability of showing heads. Let $p_1, \ldots, p_5$ denote the probability of heads in each coin. Suppose that

$$p_1 = 0; p_2 = 1/4; p_3 = 1/2; p_4 = 3/4; p_5 = 1$$

Let H denote "heads is obtained" and let C_i denote the event that coin i is selected.

(a) Select a coin at random and toss it. Suppose a head is obtained. What is the probability that coin i was selected ($i = 1, \ldots, 5$)? In other words, find $P(C_i|H); i = 1, \ldots, 5$.

(b) Toss the coin again. What is the probability of another head? In other words find $P(H_2|H_1)$ where $H_j =$ "heads on toss j".

(c) Now suppose that the experiments was carried as follows: We select a coin at random and toss until a head is obtained. Find $P(C_i|B_4)$ where $B_4 =$ "first head is obtained on toss 4".

2. Let X and Y be independent random variables and $X \sim Poisson(\theta)$ and $Y \sim Poisson(\lambda)$ where both $\theta > 0$ and $\lambda > 0$

(a) Find the distribution of $X + Y$.

(b) Find the distribution of Y given $X + Y$.

3. Suppose that X is a $N(0, 1)$ random variable so $f(x) = \frac{1}{\sqrt{2\pi}} exp(-x^2/2)$.

(a) Find $P(X^2 < 1)$ and derive the pdf of $Y = X^2$. In fact, the pdf of Y is known as a chi-square distribution with one degree of freedom.

(b) Suppose now that we have a random sample of size n, $X_1, X_2, \ldots, X_n$ of a $N(0, 1)$ distribution. Find the distribution of $\sum_{i=1}^n X_i$ and the distribution of $\sum_{i=1}^n X_i^2$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from the geometric distribution $f(x|\theta) = \theta(1 - \theta)^x; x = 0, 1, 2, \ldots$ where $0 < \theta < 1$.

(a) Does $f(x|\theta)$ belongs to the exponential family? Justify your answer.

(b) Find a minimal sufficient and complete statistic for θ.

(c) Find the method of moments estimator and the maximum likelihood estimator of θ.

(d) Is there a function of θ for which there exists an estimator with variance that coincides with the Cramer-Rao lower bound? If so find this estimator.

(e) Find the Uniformly and Minimum Variance Unbiased Estimator (UMVUE) of θ if such exists.

5. Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential(θ), and $Y_1, \ldots, Y_m$ are exponential (μ). (Z follows and exponential (β) distribution if $f(z|\beta) = \frac{1}{\beta}e^{-z/\beta}; x > 0; \beta > 0$).

   (a) Find the likelihood ratio test (LRT) of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$ and show that this test can be based on the statistic

   $$T = \frac{\sum X_i}{\sum X_i + \sum Y_i}$$

   (b) Find the distribution of T when H_0 is true.

   (c) Based on the sample $X_1, \ldots, X_n$, describe a procedure to determine a $(1 - \alpha)$ confidence interval for θ.

6. Let X and θ have a joint distribution and let $\delta(X)$ be an estimator of θ.

   (a) Show that $E(\theta|X)$ is the best estimator of θ under squared error loss in the sense that it minimizes $E_{X,\theta}[\theta - \delta(X)]^2$.

   (b) For any nonnegative function $w(\theta)$, also show that the estimate $E[w(\theta)\theta|X]/E[w(\theta)]$ minimizes the risk $E_{X,\theta}\left\{w(\theta)[\theta - \delta(X)]^2\right\}$