Statistics Masters and Ph.D. Qualifying Exam
In Class: Monday Aug 11, 2008

Instructions: The exam has 6 multi-part problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.

1. A court is investigating the possible ocurrance of an unlikely event T. The reliability of two independent witnesses called Alf and Bob is known to the court: Alf tells the truth with probability α and Bob tells the truth with probability β, and there is no collusion (agreement) between the two of them. Let A and B be the events that Alf and Bob assert (repectively) that T occurred, and let $\tau = P(T)$.

   (a) What is the probability that T occurred given that both Alf and Bob declared that T occurred?

   (b) Find the probability of a) when $\alpha = \beta = 9/10$ and $\tau = 10^{-3}$ What do you think about this value as basis for a judicial conclusion.

2. Suppose that X and Y are independent exponential random variables with parameter λ so the pdf of X is $f(x; \lambda) = \lambda exp(-\lambda x)$ and similarly for Y.

   (a) Let $W = X + Y$. Determine the distribution of W.

   (b) Let $Z = X - Y$. Show that the density of Z is

   $$f_Z(z) = \frac{\lambda}{2} \exp\{-\lambda \mid z \mid\}.$$

   (c) Find the moment generating function of Z.

3. Let the joint density for th random variables X and Y be defined by

   $$f_{X,Y}(x, y) = 6\,(1 - y), \quad 0 < x < y < 1$$

   and zero otherwise.

   (a) Evaluate $E(X)$ and $E(X^2)$ and hence find an expression for $Var(X)$, where the expectations are taken with respect to $f_X(x)$.

   (b) Derive the conditional density $f_{Y|X}(y|x)$ and the conditional expectation

   $$E_{Y|X}[(1 - Y)|X = x]$$

   for general x, $0 < x < 1$.

   (c) Evaluate $P(Y < 2X)$.

4. Let X_i, $i = 1, \ldots, n$ be *iid* with $\mathrm{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. Let $\hat{\mu}$ be given by the minimizer over c of

$$\sum_{i=1}^{n} (X_i - c)^2 + \lambda c^2.$$

   (a) Give an expression for $\hat{\mu}$

   (b) What is the mean and variance of $\hat{\mu}$?

   (c) What is the mean square error, $\mathrm{MSE}(\hat{\mu})$?

   (d) Show there is a value of λ for which $\mathrm{MSE}(\hat{\mu}) < \mathrm{MSE}(\bar{X})$.

5. Let $X_1, \ldots, X_n$ be a random sample of size $n \geq 3$ from the exponential distribution of mean $\frac{1}{\lambda}$, i.e.

$$f_X(x) = \lambda\, e^{-\lambda\, x}, \quad x > 0.$$

   (a) Find a minimal sufficient and complete statistic $T(X_1, X_2, \ldots, X_n)$ for λ.

   (b) Obtain the maximum likelihood estimator $\hat{\lambda}_n$ for λ based on the sample of size n.

   (c) Compute $E(\hat{\lambda}_n)$ and discuss if the MLE is unbiased. If it is not, find an unbiased estimator for λ.

   (d) Calculate the Cramér-Rao Lower Bound for the variance of an unbiased estimator of λ. Would you expect the bound to be attained in this example? Discuss your answer.

   (e) Compute the variance of the unbiased estimator found in part (c) and comment on its behavior for $n \to \infty$.

6. Let $X_1, \ldots, X_n$ denote the income of n people chosen at random from a certain population. Suppose that each X_i has *Pareto* density

$$f(x; \theta) = c^{\theta}\, \theta\, x^{-(1+\theta)}, \quad x > c.$$

   where $\theta > 1$ and $c > 0$. Assume c is a known constant.

   (a) Find the uniformly most powerful test for the hypothesis $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$.

   (b) Explain how you would use the central limit theorem to find an approximate rejection region and critical value of the test found in (a).