Statistics Ph.D. Comprehensive
Friday January 15, 2010

Instructions: The exam has 6 problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.

1. Let $X_1, \ldots, X_n$ be i.i.d with pdf

$$f(x|\theta, \nu) = \frac{\theta \nu^\theta}{x^{\theta+1}} I(\nu \leq x); \quad \theta > 0, \nu > 0$$

where $I(\cdot)$ denotes the indicator function.

   (a) Find a 2-dimensional sufficient statistic for the model.

   (b) Suppose θ is a known constant. Find the MLE for ν.

   (c) Now suppose that θ is unknown but $\nu = 1$. Find the score function $S(\theta)$, the derivative of the log-likelihood, and the determine its asymptotic distribution at the true value θ_0. Carefully justify your answer and clearly state any theorems that you use.

   (d) Suppose $\nu = 1$. Find the MLE for θ and determine its asymptotic distribution. Carefully justify your answer and state any theorems that you use.

   (e) Suppose $\nu = 1$. Find the asymptotic distribution of the MLE estimator of $\exp[-\theta]$. (Even if you cannot find it, show how to find it.)

2. Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$. Find the best unbiased estimator of σ^p, where p is a positive constant, not necessarily an integer. Justify why your proposed estimator is best unbiased. Hint: You can find $E(W^p)$ if $W \sim \text{Gamma}(a, b)$.

3. Find the likelihood equations and the asymptotic distribution of the MLE for the parameters of the Gamma distribution, $Ga(\alpha, \beta)$,

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-x/\beta) I(x > 0)$$

Notice that parameter space is $\Theta = \{(\alpha, \beta) : \alpha > 0; \beta > 0\}$.

4. Let $X_1, X_2, \ldots, X_n$ be a sample from the Beta distribution, $Be(\theta, 1)$, $f(x|\theta) = \theta x^{\theta-1} I(0 < x < 1)$.

   (a) Find the MLE of $1/\theta$. Show it is unbiased and achieves the Cramér-Rao lower bound.

   (b) Show $\bar{X}_n$ is an unbiased estimate of $\theta/(1+\theta)$. Compare its variance to the Cramer-Rao bound. Is $\bar{X}_n$ a UMVUE?

5. Suppose $y_1, y_2, \cdots y_n$ are iid with mean $\sqrt{\xi} - 1$ and variance ξ, and $E[(y_i - \sqrt{\xi} + 1)^4] = \xi^2$. Assume that $P(y_1 > 0) = 1$.

(a) Let $g(\xi) = ln(\sqrt{\xi})$ and $T_n = ln(1+\bar{y}_n)$. What is the limiting distribution of $\sqrt{n}(T_n - g(\xi))$?

(b) Determine up to order $1/n$ the bias of T_n as an estimator of $g(\xi)$.

(c) Consider two tests of $H_0 : \xi = 9$ vs. $H_0 : \xi > 9$.

$$T_1 : \text{Reject} \quad H_0 \quad \text{when} \quad \frac{\sqrt{n}(\bar{y}_n - 2)}{q_1} > u_\alpha$$

$$T_2 : \text{Reject} \quad H_0 \quad \text{when} \quad \frac{\sqrt{n}(T_n - ln(3))}{q_2} > u_\alpha$$

Note: $1 - \Phi(u_\alpha) = \alpha$, where Φ is the standard normal cdf. Find q_1 and q_2 so that each test has asymptotic level α.

(d) Find the asymptotic power $\lim_{n\to\infty} \beta_n(\xi_n)$ for each test, where $\xi_n = (3 + \triangle/\sqrt{n})^2 + o(1/n)$. [If $p > 0$, then $\xi_n = a_n + o(1/n)$ implies that $\xi_n^r = a_n^r + o(n^{-r})$.]

6. Consider an analysis of covariance model

$$Y = [X, Z] \begin{bmatrix} \beta \\ \gamma \end{bmatrix} + e,$$

where X is an $n \times p$ matrix, Z is an $n \times s$ matrix, $E(e) = 0$, and $Cov(e) = \sigma^2 I$. Let M be the ppo onto $C(X)$.

(a) Show that $\xi'\gamma$ is estimable if and only if $\xi' = \rho'(I - M)Z$ for some vector ρ.

(b) Assuming that $(I - M)Z$ is full rank, find the least squares estimate of γ.

(c) Assuming that $(I - M)Z$ is full rank, show that the least squares estimate of β satisfies $M\hat{\beta} = M(Y - Z\hat{\gamma})$.