Statistics Ph.D. Comprehensive
Thursday August 12, 2010

Instructions: The exam has 6 problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.

1. Suppose that X is a data vector with density $f(x|\theta)$, that $G(X)$ is any statistic, $T(X)$ is a sufficient statistic, and that $H(X)$ is a complete sufficient statistic. Suppose that $E[G(X)] = E[H(X)] = g(\theta)$.
   (a) State and prove the Rao–Blackwell Theorem
   (b) Show that $H(X)$ is the uniformly minimum variance unbiased estimate of $g(\theta)$.

2. Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$. Find the best unbiased estimator of σ^p, where p is a positive constant, not necessarily an integer. Justify why your proposed estimator is best unbiased.

3. Let $X_1, \ldots, X_n$ be a sample from
$$f(x|b,g) = \frac{1}{\Gamma(g)b^g} x^{g-1} e^{-x/b}.$$
Show that there exists a UMP test for $H_0 : b \leq b_0$ versus $b > b_0$ when g is known. Find the form of the rejection region.

4. Let $X_1, X_2, \ldots$ be independent with X_n taking the values $\sqrt{n-1}$, 1, -1, and $-\sqrt{n-1}$ each with probability 1/4. Show that $\bar{X}_n$ converges in distribution to a $N(0, .25)$.

5. In a standard linear model $Y = X\beta + e$ we know that $\hat{\beta}$ is a least squares estimate if and only if $X\hat{\beta} = MY$ where M is the perpendicular projection operator onto $C(X)$, the column space of X. Show that $\hat{\beta}$ is a least squares estimate if and only if it is a solution to the normal equations $X'X\beta = X'Y$.

6. Suppose $y_1, y_2, \cdots y_n$ are iid with mean $\sqrt{\xi} - 1$ and variance ξ, and $E[(y_i - \sqrt{\xi} + 1)^4] = \xi^2$. Assume that $P(y_1 > 0) = 1$. Let $g(\xi) = ln(\sqrt{\xi})$ and $T_n = ln(1 + \bar{y}_n)$. What is the limiting distribution of $\sqrt{n}(T_n - g(\xi))$?

7. Suppose $\epsilon_1, \epsilon_2, \ldots$ are independent random variables all having the same mean μ and variance σ^2. Define X_n as the autoregressive sequence,
$$X_1 = \epsilon_1,$$
and for $n \geq 2$,
$$X_n = \beta X_{n-1} + \epsilon_n$$
where $-1 \leq \beta < 1$. Show that $\bar{X}_n$ converges in quadratic mean to $\mu/(1-\beta)$.