Statistics Master and Ph.D. Qualifying Exam
In Class: Thursday, August 12, 2010

Instructions: The exam has 6 multi-part problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.

Problem 1. A company provides earthquake insurance. The premium X is modeled by the probability density function (p.d.f.)

$$f_X(x) = \frac{x}{5^2}\, e^{-x/5}, \quad 0 < x < +\infty$$

while the claims Y have the p.d.f.

$$g_Y(y) = \frac{1}{5}\, e^{-y/5}, \quad 0 < y < +\infty$$

If X and Y are independent, find the p.d.f. of $Z = X/Y$.

Problem 2. For the hierarchical model

$$Y|\Lambda \sim \text{Poisson}(\Lambda) \quad \text{and} \quad \Lambda \sim Gamma(\alpha, \beta)$$

find the marginal distribution, mean and variance of Y. Show that the marginal distribution of Y is a Negative Binomial when α is an integer. (Hint: If $X \sim Gamma(\alpha, \beta)$; $f(x : \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$ $, x > 0, \alpha, \beta > 0)$

Problem 3. Let X be a random variable with probability density function (p.d.f.)

$$f_X(x) = 2\, x\, e^{-x^2} I_{(0,+\infty)}(x)$$

(a) Find the distribution of $Y = X^2$ by computing its moment generating function.

(b) Let $Y_1, \ldots, Y_n$ be a random sample where the Y_i's are identically distributed as Y. Find the distribution of $Z = \sum_{i=1}^n Y_i$.

Problem 4. Let $X_1, X_2, \cdots X_n$ be a random sample with densities given by the p.d.f.

$$f(x) = \frac{1}{\theta}\, \exp(-\frac{x - \alpha}{\theta})\, I_{[\alpha,\infty)}(x)$$

for $\theta > 0$ and $-\infty < \alpha < \infty$. Suppose θ and α are unknown.

(a) Find a pair of statistics that are sufficient for (θ, α).

(b) Derive the MLE $(\hat{\theta}, \hat{\alpha})$ for (θ, α). (Hint: First find the MLE (or $\hat{\alpha}$) for α, then use $\hat{\alpha}$ to find the MLE for θ)

(c) Derive the method of moment estimators $(\tilde{\theta}, \tilde{\alpha})$ for (θ, α).

Now for parts (d) and (e) assume that $\alpha = 0$.

(d) Determine a level $1 - \lambda$ confidence interval for θ.

(e) Determine the UMVUE for $\tau(\theta) = \theta^2$.

Problem 5. Let $X_1, X_2, \cdots X_n$ be a random sample with densities given by the p.d.f. $f(x) = \dfrac{\theta}{x^{\theta+1}}$ for $\theta > 0$ and $x > 1$.

(a) Using the Neyman-Pearson lemma find the most powerful test for testing

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1$$

at level α. Assume $\theta_1 > \theta_0$.

Problem 6. Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables with densities $f(x) = x\, e^{-x} I_{(0,\infty)}(x)$. Let $\bar{X}_n = \dfrac{1}{n} \sum_{k=1}^{n} X_k$.

(a) Find the limiting distribution function of $\bar{X}_n$ as $n \to \infty$.

(b) Find the limiting distribution of $n^{\frac{1}{2}}((\bar{X}_n)^5 - 32)$ as $n \to \infty$.