Statistics Master and Ph.D. Qualifying Exam
In Class: Thursday, January 13, 2011

Instructions: The exam has 6 problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.

Problem 1. Two litters of a particular rodent species have been born, one with two brown-haired and one gray-haired (litter 1), and the other with three brown-haired and two gray-haired (litter 2). We select a litter at random and then select and offspring at random from the selected litter.

(a) What is the probability that the animal chosen is brown-haired?

(b) Given that a brown-haired offspring was selected, what is the probability that the sampling was from litter 1?

Problem 2. For the hierarchical model

$$Y|\theta \sim \text{Binomial}(n, \theta) \quad \text{and} \quad \theta \sim \text{Uniform}(0, 1)$$

assume n is fixed.

(a) Find the marginal mean and variance of Y, $E(Y)$ and $Var(Y)$.

(b) Find the marginal distribution of Y.

Problem 3. Let $X \sim N(0, 1)$ and $Y \sim N(0, 1)$. Suppose X and Y are independent and define $U = X + Y$ and $V = X - Y$.

(a) Find the joint distribution of U and V.

(b) Find the marginal distribution of U and V respectively. Justify your answer.

Problem 4. Let $X_1, X_2, \cdots X_n$ be a random sample with densities given by the p.d.f.

$$f(x) = \frac{1}{\theta} I_{[0,\theta)}(x)$$

for $\theta > 0$. Suppose that θ is unknown.

(a) Find a sufficient statistic for θ.

(b) Derive the MLE $\hat{\theta}$ for θ. Is this estimator unbiased?

(c) Derive the method of moment estimators $\tilde{\theta}$ for θ. Is this estimator unbiased?

(d) Determine a level $1-\lambda$ confidence interval for θ.

Problem 5. Let $X_1, X_2, \cdots X_n$ be a random sample from a $N(\theta, \sigma^2)$, where θ_0 is a specified value of θ and σ^2 is unknown. We are interested in testing

$$H_o : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0$$

Show that the test that rejects H_0 when

$$|\bar{X} - \theta_0| > t_{n-1,\alpha/2}\sqrt{S^2/n}$$

is a test of size α. Justify your answer.

Problem 6. Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables that are log normal (μ, σ^2), i.e., $\log(X_i) \sim N((\mu, \sigma^2)$. Find a limiting distribution as $n \to \infty$ for the geometric mean, $\tilde{X}_n = \left[\prod_{k=1}^{n} X_k\right]^{1/n}$.

Hint: Write $\tilde{X}_n$ as a simple function of a sample mean.