Statistics Master and PH.D. Qualifying Exam
In Class: 1:00pm-5:00pm, Thursday, August 11 2011

Instructions: The exam has 6 multi-part problems. All of the problems will be graded. Write your ID number on each of your answer sheets and label them. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.

Problem 1. (15) Let $X_1, \cdots, X_n$ be independent $N(0,1)$ and Let $u_4 = E[X_1^4]$ (which is greater than 1). Show that

$$\frac{\sum_{i=1}^{n}(X_i^2 - 1)}{\sqrt{n(u_4 - 1)}} \xrightarrow{d} N(0,1) \quad \text{as} \quad n \to \infty.$$

Problem 2. (15) A population having N distinct elements is sampled with replacement. Let S_r be the random sample size necessary to get r distinct elemetns ($r \leq N$). Show that

$$E[S_r] = \frac{N}{N} + \frac{N}{N-1} + \cdots + \frac{N}{N-r+1}.$$

(Hint: Write S_r as sum of r random variables)

Problem 3. (20) Let $Y_1, \cdots, Y_n$ be independent $N(0,1)$. Show that

(1) $\bar{Y}$ is independent of $Y_i - \bar{Y}$ for any i.

(2) $\bar{Y}$ is independent of $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$.

Problem 4. (25) Let $X_1, X_2, \cdots X_n$ be a random sample with densities given by the p.d.f. $f(x;\theta) = e^{-(x-\theta)}$ for $x > \theta$ and zero otherwise, where $-\infty < \theta < \infty$. Let Y_1 be the first order statistic from this random sample.

(a) Prove that Y_1 is sufficient for θ.

(b) Prove Y_1 is complete.

(c) Determine the UMVUE for $\tau(\theta) = \theta$.

Problem 5. (10) Let $X_1, X_2, \cdots X_n$ be a random sample from a $GAM(\beta, 1)$ distribution. Obtain a pivotal quantity for β and determine its distribution.

Problem 6. (15) Let X be a random sample of size 1 from a distribution with $f(x;\theta) = \theta x^{\theta-1}$ for $0 < x < 1$ and zero otherwise, where $\theta > 0$.

(a) Find a level-α UMP critical region for testing $H_0 : \theta = 1$ against $H_\alpha : \theta > 1$.

(b) Find the power function of this test.