Statistics Ph.D. Comprehensive
Wednesday, August 15, 2012

Instructions: The exam has 6 problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.

1. Consider a standard linear regression model
$$Y = X\beta + e, \qquad e \sim N(0, \sigma^2).$$
Show that the MSE is UMVU for estimating the unknown parameter σ^2.

2. Let $X_1, X_2, \ldots$ be independent with X_n taking the values $\sqrt{n-1}$, 1, -1, and $-\sqrt{n-1}$ each with probability 1/4. Find the asymptotic distribution of $\bar{X}_n$.

3. Let $X_1, X_2, \ldots$ be independent with X_n taking the values $1/\sqrt{n-1}$, 1, -1, and $-1/\sqrt{n-1}$ each with probability 1/4. Show that X_n converges in distribution to an appropriate random variable X. Show that X_n does not converge almost surely to the same X.

4. Consider testing a linear model
$$Y = X\beta + e, \qquad e \sim N(0, \sigma^2)$$
against a reduced model $Y = X_0\gamma + e$ with $C(X_0) \subset C(X)$. **In both models, σ^2 is KNOWN!** Find a monotone transformation of the generalized likelihood ratio test statistic that has an **exact small sample** χ^2 distribution under the reduced model and describe how to conduct the test.

5. Consider conducting $\alpha = 0.10$ level Neyman-Pearson (N-P) tests of $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$ for $\theta = 0, 1, \ldots, 100$. The null distribution is
$$\Pr[X = 0|\theta = 0] = .9 \quad \Pr[X = i|\theta = 0] = .001, i = 1, \ldots, 100.$$
and the alternative sampling distributions are
$$\Pr[X = 0|\theta = i] = .91 \quad \Pr[X = i|\theta = i] = .09, i = 1, \ldots, 100.$$
a) Find the generalized likelihood ratio test and its power.

   b) Find the most powerful test for $H_0 : \theta = 0$ versus $H_1 : \theta = i$ for any particular simple alternative i.

   c) If a UMP test exists, give it or explain why it does not exist.

6. Suppose that $\sqrt{n}(X_n - \mu)$ converges in distribution to a standard normal. Find an appropriate limiting distribution for nX_n^2

   a) when $\mu = 0$;

   b) when $\mu \neq 0$.