Statistics Masters and Ph.D. Qualifying Exam: In-Class

Wednesday August 15, 2012

Banner ID: ______________________________________

Instructions: *The exam has 5 multi-part problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1 (20 points):

Assume that we have observed $x_1, x_2, \ldots, x_n | \lambda \overset{iid}{\sim} Poisson(\lambda)$ and that we are interested in estimating $\tau(\lambda) = \lambda^2 e^{-\lambda}$.

(a) Find the Maximum Likelihood Estimator of $\tau(\lambda)$.

(b) Verify that

$$\hat{\tau}_{nieve} = 2I(x_1 = 2)$$

is a unbiased estimator of $\tau(\lambda)$.

(c) Find the Uniform Minimum Variance Unbiased Estimator (UMVUE) of $\tau(\lambda)$. *Hint: Rao-Blackwellize the estimator in (b).*

Problem 2 (20 points):

Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with mean of θ. Assume that the prior distribution for $\tau = \theta^{-1}$ is Gamma(α, β) where α and β are known positive constants.

(a.) What is the posterior distribution of τ?

(b.) What is the posterior mean of $\eta = e^{-\tau}$.

(c.) Find a consistent estimator of η (you must show that it is consistent).

Problem 3 (20 points):

Suppose that, in a survey of 1000 registered voters in a state, 400 say they voted in a recent primary election. Actually, though, the voter turnout was only 30% (You may assume that this is a fact).

(a.) Give an estimator and an estimate of p = probability that a nonvoter will falsely state that he or she voted. Assume that all voters honestly report that they voted.

(b.) What is the standard deviation of this estimator? Describe how you would construct a confidence interval for p.

Problem 4 (20 points):

Let X_1 and X_2 be a random sample with the following *pdf*:

$$f(x) = \begin{cases} 1.5x^2 & \text{for } -1 \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a.) Find the *cdf* and the *pdf* for $Y = X_1^2$.

(b.) Let $X_{(2)} = \max(X_1, X_2)$. Find the *pdf* for $X_{(2)}$.

(c.) What is the correlation between X_1 and $X_1 + X_2$?

(d.) Find $\text{E}[X_1X_2]$ and $\text{Var}[X_1X_2]$.

Problem 5 (20 points):

Suppose that we wish to test

$$H_0 : \theta = 3 \text{ vs. } H_1 : \theta = 2.$$

When we observe observations from the following density:

$$f(x|\theta) = \begin{cases} \theta x^{\theta - 1} & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a.) Construct the Most Powerful (MP) level-α test based on one observation $X|\theta \sim f(x|\theta)$.

(b.) What is the power of this test? *If you did not find a test in (a), consider the test which rejects when $x < .25$.*

(c.) If we have $X_1, X_2, \ldots, X_n|\theta \overset{iid}{\sim} f(x|\theta)$, what is the distribution of $T(\boldsymbol{X}) = \sum_{i=1}^{n} \log(x_i)$?

(d.) Based on this random sample of size n, what is the Uniformly Most Powerful (UMP) level-α test?