Statistics Masters and Ph.D. Qualifying Exam: In-Class

Monday January 7, 2013

Banner ID: ____________________________________

Instructions: *The exam has 6 multi-part problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1 (20 points):

Suppose X is a continuous random variable and that there exists $M > 0$ such that $P(|X| < M) = 1$.

(a) Show that $E\left(|X|\right) < M$.

(b) Show that $E\left(X\right) \in [-M, M]$.

Problem 2 (20 points):

Let $Y = (Y_1,\, Y_2)$ be a random vector with density

$$f(y_1, y_2) \;\; = \;\; \frac{2}{\pi} \exp\left[-\frac{y_1^2 + y_2^2}{2}\right] I(y_1 > 0, y_2 > 0)$$

(a) Are Y_1 and Y_2 independent? Why or why not?

(b) Let U_1, U_2 be discrete independent random variables with

(1) $P\left(U_k = -1\right) = P\left(U_k = +1\right) = \frac{1}{2}$, for $k = 1, 2$, and

(2) (U_1, U_2) independent of (Y_1, Y_2) .

Define $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \;\; = \;\; \begin{bmatrix} U_1 \cdot Y_1 \\ U_2 \cdot Y_2 \end{bmatrix}$.

Show that Z has a standard bivariate normal distribution.

(c) Show that $Y_1^2 + Y_2^2$ has a χ^2 distribution with 2 degrees of freedom.

Hint: *A random variable W with the standard half-normal distribution has density $f_W(w) = \sqrt{\frac{2}{\pi}} \exp\left[-\frac{w^2}{2}\right] I(w \geq 0)$.*

Problem 3 (20 points):

Let X and Y be two random variables with

- $X \sim$Gamma(k_1, γ), i.e. a Gamma distribution with shape parameter k and scale parameter γ,
- $Y \sim$Gamma(k_2, γ),
- X, Y are independent.

(a) Show that $Z = X + Y \sim$Gamma$(k_1 + k_2, \gamma)$.

(b) Find the conditional density of X given Z.

(c) Conclude that $\frac{X}{Z} \sim$Beta(k_1, k_2).

Problem 4 (20 points):

Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$.

(a) Find the best unbiased estimator for σ^2. Justify your answer.

(b) Give the sampling distribution of the estimator found on part (a).

(c) Now find the best unbiased estimator of σ^p, where p is a positive constant, not necessarily an integer. Justify why your proposed estimator is best unbiased.

Problem 5 (20 points):

Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \frac{1}{\theta}, 0 \leq x \leq \theta, \theta > 0$$

(a) What is a minimal sufficient and complete statistic for θ? Justify your answer.

(b) Estimate θ using both the method of moments and maximum likelihood. Calculate means and variances of both estimators. Which estimator should be preferred and why?

(c) Propose one method to estimate a confidence interval for θ and then derive a $(1-\alpha)$ confidence interval for θ.

Problem 6 (20 points):

Let $f(x|\theta)$ be the logistic location pdf

$$f(x|\theta) = \frac{e^{(x-\theta)}}{\left(1+e^{(x-\theta)}\right)^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

(a) Show that this family has a monotone likelihood ratio.

(b) Based on one observation, X, find the most powerful α test of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$. The rejection region must have an explicit form in terms of X and α.

(c) Show that the test in part (b) is Uniformly Most Powerful (UMP) size α for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$.