Statistics Masters and Ph.D. Qualifying Exam: In-Class

Monday August 12, 2013

Banner ID: ________________________________

Instructions: *The exam has 5 multi-part problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1 (30 points):

Let X be a random variable with probability density function (*pdf*) given by

$$f_X(x|\theta) = \begin{cases} \frac{\theta}{2}e^{-\theta|X|} & \text{for } -\infty < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

where $\theta > 0$ is an unknown parameter.

(a.) Find the cumulative distribution function (*cdf*) for the random variable $Y = \theta|X|$ and identify the distribution.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n > 2$ from $f_X(x|\theta)$.

(b.) Find the maximum likelihood estimator, $\hat{\theta}_{mle}$, for θ.

(c.) Find the Uniform Minimum Variance Unbiased Estimator (UMVUE), $\hat{\theta}_{umvue}$, for θ.

(d.) Which estimator, $\hat{\theta}_{mle}$ or $\hat{\theta}_{umvue}$, do you prefer in terms of Mease Squared Error (MSE)?

(e.) Which of your estimators are consistent? Do not simply quote a result about the consistency of MLE's.

(f.) Construct a 95% Confidence Interval for θ.

Problem 2 (20 points):

Consider a list of m names, where the same name may appear more than once on the list. Let $n(i)$ denote the number of times that the name in position i appears on the list, $i = 1, 2, \ldots, m$. Let d denote the number of distinct names on the list.

(a.) Express d in terms of the variables m and $n(i)$, $i = 1, 2, \ldots, m$.

(b.) Let $U \sim Uniform(0, 1)$ and let $X = \lfloor mU \rfloor + 1$. Where $\lfloor \cdot \rfloor$ denotes the *floor function*, which rounds down to nearest integer. Ex: $\lfloor 1.1 \rfloor = 1, \lfloor 1.5 \rfloor = 1, \lfloor 1.99 \rfloor = 1, \lfloor 2 \rfloor = 2$. What is the probability mass function (pmf) of X.

(c.) Show that $\mathrm{E}\left[\frac{m}{n(X)}\right] = d$.

Problem 3 (20 points):

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f$. Where, $\mathrm{E}\left[X_i^j\right] = \theta_j < \infty$ for all $j = 1, 2, \ldots, \infty$.

(a.) State and prove a Weak Law of Large Numbers.

(b.) What is the limiting distribution of

$$\sqrt{n}\left(\bar{X}_n^2 - \theta_1^2\right)$$

(c.) Find an a and b such that

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - a\right) \xrightarrow{d} N(0, b)$$

Problem 4 (20 points):

Let $\boldsymbol{X} = (X_1, X_2)$ be a bivariate random variable from the following joint *pdf*.

$$f(\boldsymbol{X}) \propto \frac{1}{\sqrt{X_1 X_2}} I(0 < X_1 < X_2 < 1)$$

(a.) Verify that f is a valid joint *pdf*.

(b.) Find $P(2X_1 > X_2)$.

(c.) Find the conditional density $f(X_2|X_1)$.

(d.) Find the marginal density $f(X_1)$.

(e.) What is $\mathrm{E}\left[X_2\right]$

Problem 5 (20 points):

Let $X_1, X_2, \ldots, X_n|\theta \overset{iid}{\sim} f(x|\theta)$ where

$$f(x|\theta) = \begin{cases} \frac{\theta}{10}\left(\frac{x}{10}\right)^{\theta-1} & \text{if } 0 < x < 10 \\ 0 & \text{elsewhere} \end{cases}$$

(a.) Find the form of the UMP-level-α test for testing

$$H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta = \theta_1$$

for $\theta_1 > \theta_0 > 0$.

(b.) Find the form of the UMP-level-α test for testing

$$H_0 : \theta \leq \theta_0 \text{ vs. } H_1 : \theta > \theta_1$$

for $\theta_1 > \theta_0 > 0$.