Statistics Masters and Ph.D. Qualifying Exam: In-Class

Tuesday, January 14, 2014

Banner ID: ___________________________________

Instructions: *The exam has 5 multi-part problems. All of the problems will be graded. Write your ID number on your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1 (40 points):

Assume that we have observed $x_1, x_2, \ldots, x_n | p \overset{iid}{\sim} Binomial(m, p)$ and that we are interested in estimating $\tau(p) = (1-p)^m$ and m is a known constant. Recall that the probability mass function (pmf) of a $Binomial(m, p)$ is

$$f(x|m,p) = \begin{cases} \binom{m}{x} p^x (1-p)^{m-x} & \text{if } x = 0, 1, \ldots, m \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the Maximum Likelihood Estimator of $\tau(p)$.

(b) Verify that

$$\hat{\tau}_{naive} = I(x_1 = 0) = \begin{cases} 1 & \text{if } x_1 = 0 \\ 0 & \text{elsewhere} \end{cases}$$

is a unbiased estimator of $\tau(p)$.

(c) Show that $T = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for p.

(d) Show that $T | X_1 = 0 \sim Binomial((n-1)m, p)$

(e) Find the Uniform Minimum Variance Unbiased Estimator (UMVUE) of $\tau(p)$. *Hint: Rao-Blackwellize the estimator in (b).*

Problem 2 (20 points):

A researcher is interested in studying illegal drug usage in school. Because people often lie about drug usage, the following scheme was used (based off of Warner, 1965). The researcher had each participant flip a fair coin (the researcher was not informed of the results of the flip). After the flip, the researcher asked the respondent to answer yes or no.

- If the coin landed head: I used an illegal drug in the past month.
- If the coin landed tails: My birthday was in the last month.

Define θ as the proportion of all students who used drugs in the past month and $p = \frac{1}{12}$ is the proportion of all students who were born in the last month.

(a.) Let Y_i be a random variable which indicates if the person answered yes ($y_i = 1$) or no ($y_i = 0$) to the question. One way to represent Y_i is as follows

$$Y_i = Z_i D_i + (1 - Z_i) B_i$$

Where $Z_i \sim Bernoulli(.5)$ indicates if the person flipped a heads, $D_i \sim Bernoulli(\theta)$ indicates if the person used illegal drugs in the last month, and $B_i \sim Bernoulli(p)$ indicates if the person was born in the last month. Note that only Y_i is observed of these random variables.

Find the mean and variance of Y_i conditional on θ. You may assume that Z_i, D_i and B_i are independent.

(b.) Show that $\hat{\theta} = 2\frac{\sum_{i=1}^{n} Y_i}{n} - p$ is an unbiased estimator of θ. You may use the fact that $Y_i|\theta \overset{iid}{\sim} Bernoulli(.5\theta + .5p)$.

(c.) Find the asymptotic distribution of $\hat{\theta}$.

Problem 3 (10 points):

Suppose X is one observation from a Binomial$(5, \theta)$ distribution where $0 < \theta < 1$.

(a.) If $\pi(\theta)$, the prior distribution for θ, is a Beta(1,1) distribution, find the posterior distribution for θ, $\pi(\theta|x)$. What is the posterior mean of θ?

(b.) Suppose we wish to test $H_0 : \theta = 1/2$ versus $H_1 : \theta = 3/4$. What is the rejection region corresponding to the uniformly most powerful (UMP) level-α test? Justify your answer.

Problem 4 (10 points):

Suppose $X_1, \ldots, X_n$ are independent, identically distributed from the uniform distribution on $(0, \theta)$ for unknown $\theta > 0$ and $T = max\{X_1, X_2, \ldots, X_n\}$.

(a.) Show that T/θ is a pivotal quantity. Justify your answer.

(b.) Based on the previous item, give an expression for a $(1 - \alpha)$ level confidence interval for θ where $0 < \alpha < 1$.

Problem 5 (20 points):

Suppose $X_1, \ldots, X_n$ are a random sample from the $Gamma(\alpha, \beta)$ for unknown $\alpha > 2$ and $\beta > 0$. *Note: use the Gamma distribution with mean equal to $\alpha\beta$ and variance equal to $\alpha\beta^2$.*

(a.) Find a and b such that

$$\sqrt{n}\left(\frac{n}{\sum_{i=1}^n X_i} - a\right) \xrightarrow{d} N(0, b).$$

(b.) Verify that

$$\mathrm{E}\left[\frac{1}{X^k}\right] = \frac{\Gamma(\alpha - k)}{\Gamma(\alpha)\beta^k}$$

if $X \sim Gamma(\alpha, \beta)$ and $k > \alpha$.

(c.) Let $Y_i = \frac{1}{X_i}$. Find a and b such that

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n Y_i - a\right) \xrightarrow{d} N(0, b).$$