In-Class Statistics Masters and Ph.D. Qualifying Exam

August, 2014

Instructions: *The exam has 7 multi-part problems. All of the problems will be graded. Write your ID number on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. (10 pts) Let $X_1, X_2, \ldots$ be a sequence of independent random variables, each uniformly distributed on the interval of $(0, 1)$. Let

$$Y = X_1^a + (X_1 X_2)^a + (X_1 X_2 X_3)^a + \cdots ,$$

where $a > 0$, find the mean of the random variable Y.

Problem 2. (10 pts) A fair six sided die is rolled repeatedly until all 6 faces appear at least once. Let X denote the number of rolls necessary. Find the mean and standard deviation of X.

Problem 3. (18 pts) Let $X_i, i = 1, 2, \cdots$, be iid with density function

$$f(x) = \begin{cases} 3(1-x)^2, & \text{for } 0 < x < 1 \\ 0 & \text{else.} \end{cases}$$

and let $Y_n = \min(X_1, X_2, \cdots, X_n)$.

(a) Find the cdf of Y_n, $F_n(y) = P(Y_n \leq y)$.

(b) Find the constant c, such that $Y_n \xrightarrow{P} c$, and verify the convergence.

(c) Show that $D_n = nY_n$ converges in distribution and find the limiting distribution.

Problem 4. (16 pts) Suppose that $X_1, \cdots, X_n$ is a random sample from the probability density function

$$f(x|\theta) = \frac{rx^{r-1}}{\theta} e^{-x^r/\theta}, 0 < x < \infty,$$

where $r > 1$ is a known constant and $\theta > 0$ is an unknown parameter.

(a) Find the MLE of θ.

(b) Based on the MLE of θ, find an unbiased estimator of 2θ.

Problem 5. (18 pts) Suppose the distribution of Y, conditional on $X = x$ is $N(x, x^2)$ and that the marginal distribution of X is uniform $(0, 10)$.

(a) Find the expected value of Y, $E(Y)$.

(b) Find the variance of Y, $V(Y)$.

(c) Find the covariance between X and Y, $\text{cov}(X, Y)$.

(d) Prove that Y/X and X are independent.

Problem 6. (18 pts) Let $X_1, \cdots, X_n \overset{iid}{\sim} f(x|\theta) = \dfrac{x^2}{\theta^3}\sqrt{\dfrac{2}{\pi}} exp\left(-\dfrac{x^2}{2\theta^2}\right)$ for $x \geq 0$ and $\theta > 0$. Note that

$$E(X_1) = 2\theta\sqrt{\frac{2}{\pi}}, E(X_1^2) = 3\theta^2, E(X_1^3) = 8\theta^3\sqrt{\frac{2}{\pi}}, E(X_1^4) = 15\theta^4$$

(a) Find a complete sufficient statistic for θ.

(b) Find an uniformly minimum variance unbiased estimator (UMVUE) for θ^2.

(c) Does the estimator in part (b) achieve the Cramer-Rao lower bound (CRLB) for estimating θ^2?

Problem 7. (10 pts) Suppose that $X_1, X_2, \cdots, X_{100}$ are a random sample of size 100 from a normal distribution with mean θ and variance 25.

(a) use the Neyman-Pearson lemma to find the most powerful test of size $\alpha = 0.05$ for testing $H_0 : \theta = 70$ against $H_1 : \theta = 72$.

(b) compute the power of the most powerful test of size $\alpha = 0.05$ for testing $H_0 : \theta = 70$ against $H_1 : \theta = 72$.