In-Class Statistics Masters and Ph.D. Qualifying Exam

Jan, 2015

Instructions: *The exam has 5 multi-part problems. All of the problems will be graded. Write your ID number on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. (10 pts) Two numbers are independently and uniformly chosen from the interval (0,1). What is the probability that the sum of the numbers is less than 1 and the product of the numbers is less than $\frac{2}{9}$? (Note that both conditions hold simultaneously.)

Problem 2. (10 pts)

A fair die is tossed repeatedly. Let X be the number of tosses until the number 5 appears for the first time, and let Y be the number of tosses until the number 5 appears for the second time. Find $E[X|Y]$, the conditional expectation of X given Y.

Problem 3. (15 pts)

$X_1, X_2, \ldots$ form an iid sequence of uniform (0,1) random variables. Let c be a fixed number in (0,1). Let $N = \text{minimum}\{n : X_n > c\}$.

(a) Find $P(N = n)$, the probability mass function of N.

(b) Find the distribution function of X_N given $N = n$.

(c) Find the distribution function of X_N.

Problem 4. (15 pts) X and Y are independent, standard normal random variables. Determine the conditional distribution of X given that $X + Y = v$.

Problem 5. (50 pts) Let $X_1, X_2, \ldots, X_n$ be a random sample of independent Poisson random variables with mean μ.

(a) (10 pts) Let $\theta = P(X_1 = 0) = e^{-\mu}$. Find the MLE $\hat{\theta}$ of θ.

(b) (10 pts) Is $\hat{\theta}$ an unbiased estimator of θ? Justify your answer.

(c) (5 pts) Show that $\tilde{\theta} = \left[\frac{n-1}{n}\right]^{\sum_{k=1}^{n} X_k}$ is an unbiased estimator of θ.

(d) (10 pts) Find the Cramer-Rao Lower Bound (CRLB), based on the random sample $X_1, X_2, \ldots, X_n$, for the variances of unbiased estimators of θ.

(e) (10 pts) Show that $\mathrm{Var}(\tilde{\theta})$ is not equal to the CLRB obtained in (d), but $\tilde{\theta}$ is a UMVUE of θ.

(f) (5 pts) Determine the critical region of the uniformly most powerful test (UMP) of size α for $H_0 : \mu \geq \mu_0$ versus $H_\alpha : \mu < \mu_0$.