In-Class Statistics Masters and Ph.D. Qualifying Exam

August 9, 2016, 9:00am-1:00pm

Instructions: *The exam has 5 multi-part problems. All of the problems will be graded. Write your code words on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. (25 pts) X and Y have a joint probability mass function

$$P(X = x, Y = y) = e^{-(a+bx)} \left(\frac{a^x}{x!}\right) \frac{(bx)^y}{y!}$$

where $x = 0, 1, 2, \cdots, y = 0, 1, 2, \cdots$

(a) Find $E[Y|X]$.

(b) Find Var(Y).

Problem 2. (25 pts) X and Y are independent and uniformly distributed on the interval $(0, 1)$. If $U = X + Y$, and $V = \dfrac{X}{Y}$, find the joint density for U and V and the marginal densities for U and V.

Problem 3. (30 pts) Let $W_i, i = 1, \cdots, n$ be iid $\text{Exp}(\theta_1)$, $V_i, i = 1, \cdots, m$ be iid $\text{Exp}(\theta_2)$, and two samples are independent,

$$f_{W_i}(w) = \frac{1}{\theta_1} e^{-\frac{w}{\theta_1}}, \ f_{V_i}(v) = \frac{1}{\theta_2} e^{-\frac{v}{\theta_2}}$$

Recall that if $Y \sim \text{Exp}(\theta)$, then $E(Y) = \theta$ and $\text{Var}(Y) = \theta^2$.

(a) Construct a $(1-\alpha)$ confidence interval formula for θ_1/θ_2 using all the data.

(b) Find a set of sufficient statistics for (θ_1, θ_2).

(c) Provide the MLE for $\tau(\theta_1, \theta_2) = (\theta_1 - \theta_2)^2$.

(d) If a UMVUE exists for $(\theta_1 - \theta_2)^2$, provide it; otherwise explain why it does not exist.

(e) Construct a Generalized Likelihood Ratio test of the hypothesis:

$$H_0 : \theta_1 = 2\theta_2 \text{ vs. } H_\alpha : \theta_1 \neq 2\theta_2.$$

Provide the test using the exact distribution of the test statistic.

Problem 4. (10 pts) Consider the model

$$Y_i \overset{iid}{\sim} Bernoulli(p_i) \text{ for } i = 1, \cdots, n$$

and with

$$p_i = e^{\beta c_i}/(1 + e^{\beta c_i})$$

where c_is are constants.

(a) Find a sufficient statistic for β.

(b) Give the form of the uniformly most powerful test (UMP) of $H_0 : \beta = 0$ vs. $H_\alpha : \beta > 0$ by first finding a statistic T with the monotone likelihood ratio property (MLR).

Problem 5. (10 pts) Let $X_1, X_2, \ldots, X_n$ be independent, exponential random variables of mean θ.

(a) Find a minimal sufficient statistics for θ and its maximum likelihood estimator.

(b) Show that $Y = 2\sum_{i=1}^{n} X_i/\theta$ is a pivotal quantity, and construct a $(1 - \alpha)$-level confidence interval for θ.