In-Class Statistics Masters and Ph.D. Qualifying Exam

January 2017

Instructions: *The exam has 5 multi-part problems. All of the problems will be graded. Write your code words on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. Let X be a normal random variable with mean 0 and variance 1 and let Y be uniform(0,1) with X and Y being independent. Let $U = X + Y$ and $V = X - Y$. For this problem recall the density for a normal random variable is

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

(a) Find the joint distribution of U and V.

(b) Find the marginal distributions of U and V.

(c) Find $Cov(U, V)$.

Problem 2. Let $X_1, \ldots, X_n$ be an i.i.d. sample from the density

$$f(x|\theta) = \frac{1}{2\theta} e^{-|x|/\theta}, \quad -\infty < x < \infty$$

(a) Find a M.O.M. estimate of θ.

(b) Find the MLE of θ.

(c) Find a minimal, complete, sufficient statistic for θ.

(d) Show that the MLE for θ obtains the Cramer-Rao lower bound.

Problem 3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from an exponential distribution with mean θ_1 and let $Y_1, \ldots, Y_n$ be i.i.d. random variables from an exponential distribution with mean θ_2. Assume that the two samples are independent. Recall that an exponetial random variable with mean θ has density

$$f(x|\theta) = \begin{cases} 0 & x < 0 \\ \frac{1}{\theta} e^{-x/\theta} & 0 \le x < \infty \end{cases}$$

(a) Assuming that $\theta = \theta_1 = \theta_2$, find the MLE of θ when $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are observed.

(b) Find the LRT to test the hypothesis that $\theta_1 = \theta_2$ versus $\theta_1 \neq \theta_2$.

(c) Assuming again that $\theta = \theta_1 = \theta_2$, find an expression for a 95% confidence interval for θ. In the case when $\theta_1 \neq \theta_2$, would it be possible to find an approximate 95% confidence interval for $\tau = \theta_1 - \theta_2$? If so, find the interval.

Problem 4. Let X be binomial with parameters n and p. Note that for an event occurring with probability p, the odds of the event occurring are given by the expression $p/(1-p)$.

(a) Show that

$$E\left[\frac{X/n}{1-X/n}\right] \geq \frac{p}{1-p}$$

(b) Use the delta method to find approximate formulas for $E\left[\frac{X/n}{1-X/n}\right]$ and $Var\left(\frac{X/n}{1-X/n}\right)$.

(c) Suppose that drivers who don't wear seat belts are three times as likely to die as those who do wear a seat belt when there is a car accident. Suppose the odds of a randomly selected driver wearing a seat belt are 9.0 (i.e., 9 to 1 odds in favor of wearing a seat belt). What are the odds that the driver was wearing a seat belt given that the driver died in a car crash?

Problem 5. A lopsided six-sided die is rolled repeatedly, with each roll being independent. The probability of rolling the value i is p_i, $i = 1, \ldots, 6$. Let X_n denote the number of distinct values that appear in n rolls.

(a) Find $E[X_n]$ and $E[X_n^2]$.

(b) What is the probability that in the n rolls of the dice, for $n \geq 3$, a 1, 2, and 3 are each rolled at least once?