Statistics comprehensive exam. August 2017

Instructions: *The exam has 6 problems. All parts of all problems will be graded. Write your code words on each of your answer sheets. Do not put your name or UNM ID on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

1. For the linear model $Y = X\beta + e$, $E(e) = 0$, $Cov(e) = \sigma^2 V$ where V is positive definite, the normal equations for finding the generalized least squares estimate of β are $X'V^{-1}X\beta = X'V^{-1}Y$. The generalized least squares estimates also solve $X\hat{\beta} = AY$ where $A \equiv X\left(X'V^{-1}X\right)^{-}X'V^{-1}$ is an oblique projection operator onto the column space $C(X)$.

   Now consider the partitioned model $Y = X\beta + Z\gamma + e$, $E(e) = 0$, $Cov(e) = \sigma^2 V$ and the alternative model $Y = X\delta + (I - A)Z\gamma + e$ where A is exactly as defined earlier.

   (a) Show that $C(X, Z) = C[(X, (I - A)Z]$.

   (b) Show that γ is the same vector in both partitioned models.

   (c) Show how to estimate δ and γ by solving the normal equations for the alternative model.

   (d) Show how to estimate β using the estimates of δ and γ.

2. Let $X_1, X_2, \ldots$ be independent with X_n taking the values $\sqrt{n-1}$, 1, -1, and $-\sqrt{n-1}$ each with probability 1/4. Show that $\bar{X}_n$ converges in distribution to a $N(0, .25)$.

3. Suppose X_n is Binomial with n trials and probability of success p.

   (a) After suitable normalization, find the asymptotic distribution of the estimated probability, $\hat{p}_n \equiv X_n/n$.

   (b) After suitable normalization, find the asymptotic distribution of the estimated odds, $\hat{O}_n \equiv \hat{p}_n/(1 - \hat{p}_n)$.

4. Consider the balanced one-way ANOVA model $y_{ij} = \mu_i + e_{ij}$, with independent errors that are $N(0, \sigma^2)$, $i = 1, \ldots, a$, $j = 1, \ldots, N$.

   (a) Find the MLEs of the μ_is and σ^2.

   (b) Find the joint asymptotic distribution of the μ_is and σ^2 for a fixed and $N \to \infty$. (You can use $a = 2$ for this part.)

   (c) Show that for N fixed and $a \to \infty$ the MLE of σ^2 is not consistent.

5. Suppose $X|p \sim \text{Bin}(n, p)$ and $p \sim \text{Beta}(\alpha, \beta)$.

   (a) Find $E(p|X)$.

   (b) Find $R[p, E(p|X)] \equiv E_{X|p}\{[p - E(p|X)]^2\}$. Hint: This is a quadratic in p.

   (c) Find α and β so that $E(p|X)$ is an equalizer rule, i.e., the risk does not depend on p.

   (d) Find the least favorable prior distribution and the minimax decision rule.

6. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables (i.i.d.) with the probability density function,

$$f(x|\mu, \sigma) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right); x > \mu$$

   and 0 otherwise, where $-\infty < \mu < \infty$ and $\sigma > 0$ are unknown parameters.

   (a) Identify a set of sufficient statistics for the unknown parameters (μ, σ^2).

   (b) Derive the maximum likelihood estimator (MLE) of (μ, σ^2).

   (c) Suppose that $\mu = 0$ is known. Derive the UMP test for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$ for two fixed parameter values $\sigma_1 > \sigma_0 > 0$ at the $\alpha > 0$ level. Specify the rejection region.