In-Class Statistics Masters and Ph.D. Qualifying Exam

January 2018

Instructions: *The exam has 5 multi-part problems. All of the problems will be graded. Write your code words on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. Let X be a uniform$(0,1)$ random variable and let Y be uniform(1,2) with X and Y being independent. Let $U = X/Y$ and $V = X$.

(a) Find the joint distribution of U and V.

(b) Find the marginal distributions of U.

Problem 2. Suppose a website sells X computers where X is modeled as a geometric random variable with parameter p_1. Suppose that each computer is defective (i.e., needs to be returned for repair or replacement), independently with probability p_2. Let Y be the number of computers sold which are defective. For this problem, recall that a geometric random variable X with parameter p_1 has pmf

$$P(X = i) = \begin{cases} (1-p_1)^{i-1}p_1 & i = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E[Y]$.

(b) Find $Var(Y)$.

(c) Find $P(Y = 0)$.

Problem 3. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a geometric distribution with parameter p.

(a) Define U as

$$U = \begin{cases} 1 & X_1 = 1 \\ 0 & X_1 > 1 \end{cases}$$

Find $E[U]$.

(b) Find a sufficient statistic T for p.

(c) Explain why U is not a sufficient statistic.

(d) Calculate $E[U|T]$.

(e) Find an UMVUE for p.

Problem 4. Consider testing $H_0 : f(x) = 2x\, I(0 < x < 1)$ against $H_1 : f(x) = 3x^2\, I(0 < x < 1)$ with n i.i.d. observations at level $\alpha = .10$. Describe the rejection region for the most powerful test.

Problem 5. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a uniform$(\theta - 0.5, \theta + 0.5)$ distribution, where θ is unkown.

(a) Find a method of moments estimate of θ.

(b) Suppose $n = 2$ and the data are

$$0.6, 0.9$$

Find a formula for the likelihood function, and also sketch the likelihood function.

(c) Note that when there are n observations, the maximum likelihood function does not have a unique maximum. Show that one possible maximum is the midrange:

$$\widehat{\theta} = \frac{X_{(1)} + X_{(n)}}{2}$$

(d) Find the mean squared errors for the method of moments estimator and midrange.

(e) Suppose that a confidence interval for θ is constructed as (L, U) using

$$L = \overline{X}_n - 1/n, \quad U = \overline{X}_n + 1/n$$

Supposing n is large, find a formula for the approximate confidence level of the interval. Recall that for a uniform(0,1) random variable, the variance is 1/12.