Statistics comprehensive exam. August 2018

Instructions: *The exam has 5 (sometimes multi-part) problems. All of the problems will be graded. Write your code words on each of your answer sheets. Do not put your name or UNM ID on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

1. Suppose that X is a data vector with density $f(x|\theta)$, that $G(X)$ is any statistic, $T(X)$ is a sufficient statistic, and that $H(X)$ is a complete sufficient statistic. Suppose that $E[G(X)] = E[H(X)] = g(\theta)$.

   (a) State and prove the Rao–Blackwell Theorem. Hint: This has nothing to do with $H(X)$.

   (b) Explain why $T(X)$ has to be sufficient.

   (c) Show that $H(X)$ is the uniformly minimum variance unbiased estimate of $g(\theta)$.

2. Let $X_1, \ldots, X_n$ be iid $N(\mu.\sigma^2)$. Find the best unbiased estimator of σ^p, where p is a positive constant, not necessarily an integer. Justify why your proposed estimator is best unbiased. Hint: You can find $E(W^p)$ if $W \sim \text{Gamma}(a, b)$.

3. Let $X_1, \ldots, X_n$ be i.i.d with pdf

$$f(x|\theta, \nu) = \frac{\theta \nu^\theta}{x^{\theta+1}} I(\nu \leq x); \quad \theta > 0, \nu > 0$$

   where $I(\cdot)$ denotes the indicator function.

   (a) Find a 2-dimensional sufficient statistic for the model.

   (b) Suppose θ is a known constant. Find the MLE for ν.

   (c) Now suppose that θ is unknown but $\nu = 1$. Find the score function $S(\theta)$, the derivative of the log-likelihood, and the determine its asymptotic distribution at the true value θ_0. Carefully justify your answer and clearly state any theorems that you use.

   (d) Suppose $\nu = 1$. Find the MLE for θ and determine its asymptotic distribution. Carefully justify your answer and state any theorems that you use.

   (e) Suppose $\nu = 1$. Find the asymptotic distribution of the MLE estimator of $exp[-\theta]$.

4. Let $X_1, X_2, \ldots$ be independent with X_n taking the values $\sqrt{n-1}$, 1, -1, and $-\sqrt{n-1}$ each with probability 1/4. Show that $\bar{X}_n$ converges in distribution to a $N(0, .25)$.

5. Consider a linear model

$$Y = X\beta + e, \qquad \text{E}(e) = 0,$$

with least squares estimates $\hat{\beta}$ and residuals $\hat{e} = Y - X\hat{\beta}$. We are interested in the least squares estimate of γ in the ACOVA model

$$Y = X\eta + Z\gamma + e, \qquad \mathrm{E}(e) = 0.$$

The stagewise estimate is defined as $\tilde{\gamma}$, the least squares estimate from $\hat{e} = Z\gamma + \xi$.

(a) When does the stagewise estimate agree with the least squares ACOVA estimate?

(b) Find a matrix W such that fitting $\hat{e} = W\gamma + \xi$ gives the least squares ACOVA estimate.

(c) What modification is needed to the MSE from $\hat{e} = W\gamma + \xi$ to get the ACOVA model MSE?