In-Class Statistics Masters and Ph.D. Qualifying Exam

August 2018

Instructions: *The exam has 7 multi-part problems. All of the problems will be graded. Write your code words on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1 (10pts). A coin is tossed and has three possible outcomes: Head, Tail, Edge. Suppose $P(Head) = 1/3, P(Tail) = 1/2, P(Edge) = 1/6$. The coin is repeatedly tossed. If the trials are identical and independent, what is the probability of getting a head before getting a tail?

Problem 2 (15pts). Consider the following joint density function

$$f(x,y) = \begin{cases} x^2 e^{-x(y+1)} & x, y \geq 0 \\ 0, & \text{else} \end{cases}$$

(a) Find the conditional density function of Y given X.

(b) Find $E(Y|X)$.

(c) Find $\mathrm{Var}(Y|X)$.

Problem 3 (15pts). Let X and Y have joint pdf

$$f(x,y) = \begin{cases} 3x & 0 < y < x < 1 \\ 0, & \text{else} \end{cases}$$

Find the pdf of $Z = X + Y$.

Problem 4 (10pts). A piece of equipment has 5 independent parts at least 4 of which must remain active in order for the equipment to function. The lifetime (in hours) of each component follows a distribution with probability density function

$$f(x) = \begin{cases} \dfrac{1}{20} e^{-x/20}, & x > 0 \\ 0, & \text{else.} \end{cases}$$

Find the probability density function of the lifetime T (in hours) of the equipment.

Problem 5 (15pts). Suppose that we observe a random sample $X_1, \ldots, X_n$ from the density

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} m x^{m-1} e^{-x^m/\theta}, & x \geq 0 \\ 0, & \text{else,} \end{cases}$$

where m is a known constant which is greater than zero, and $\theta > 0$.

(a) Find the most powerful test for testing
$H_0 : \theta = \theta_0$ against
$H_\alpha : \theta = \theta_1 (\theta_1 > \theta_0)$

(b) Indicate how you would find the power of the most powerful test when $\theta = \theta_1$. (**Do not perform any calculation**).

(c) Is the resulting test uniformly most powerful for testing $H_0 : \theta = \theta_0$ against $H_\alpha : \theta > \theta_0$? Explain why or why not.

Problem 6 (15pts). Let X denote the number of days until failure of a water pump at a nuclear power plant. An engineer at the plant assumes that X has a geometric discrete density

$$f(x,\theta) = \begin{cases} (1-\theta)^x \theta, & x = 0, 1, 2, \cdots \\ 0, & \text{else,} \end{cases}$$

where $\theta = P(X = 0), \theta \in (0, 1)$. One can show that

$$E(X) = \frac{1-\theta}{\theta}.$$

Suppose that $\mathbf{X} = (X_1, \cdots, X_n)$ is a random sample of failure times and that the engineer's assumption is correct. Let $S = \sum_{i=1}^n X_i$. It can be shown that S has the negative binomial probability function

$$f(s;\theta) = \left(\begin{array}{c} n+s-1 \\ s \end{array} \right) \theta^n (1-\theta)^s, s = 0, 1, 2, \cdots$$

(a) Show that S is a complete sufficient statistic for θ. Based on X, what is the uniformly minimum variance unbiased estimate of $\frac{1-\theta}{\theta}$?

(b) Give an unbiased estimate of θ based on X_1.

(c) Give the uniformly minimum variance unbiased estimate of θ based on $\mathbf{X}$.

Problem 7 (20pts). Let $F(x)$ be a cumulative distribution function (cdf) with corresponding pdf $f(x)$, i.e.,

$$f(x) = \frac{d}{dx} F(x).$$

We assume F and f are known, but we don not specify what they are. Let $X_1, \cdots, X_n$ be independent and identically distributed with cdf

$$P(X_i \leq x_i) = F^\theta(x_i) = [F(x_i)]^\theta, \theta \in \Omega = (0, \infty).$$

(a) Find a maximum likelihood estimator (MLE) of θ.

(b) Define $v = \theta^{-1}$. Give the MLE for v.

(c) Find the Cramer-Rao lower bound for unbiased estimators of v

(d) Give a large sample confidence interval for v. Justify your result.