In-Class Statistics Masters and Ph.D. Qualifying Exam

January 2019

Instructions: *The exam has 7 multi-part problems. All of the problems will be graded. Write your code words on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1 (10 pts). Let X have the density

$$f(x) = \frac{2}{9}(x+1), -1 \le x \le 2.$$

- (a) Find the density of $Y = X^2$.
- (b) Find a monotone function $u(x)$ such that the random variable $U(X)$ has a uniform $(0,1)$ distribution.

Problem 2 (10 pts). Let X and Y be independent $N(0,1)$ random variables, and define a new random variable Z by

$$Z = \begin{cases} X & \text{if } XY > 0 \\ -X & \text{if } XY < 0 \end{cases}$$

Find the distribution of Z.

Problem 3 (15 pts). An employer is about to hire one new employee from a group of N candidates, whose future potential can be rated on a scale from 1 to N and a larger number means greater potential. The employer proceeds according to the following rules:

- (a) Each candidate is seen in succession (in a random order) and a decision is made whether to hire the candidate.
- (b) Having rejected $m-1$ candidates, the employer can hire the mth candidate only if the mth candidate is better than the previous $m-1$.

Suppose a candidate is hired on the ith trial, what is the probability that the j-th best candidate ($j \ge i$) was hired?

Problem 4 (15 pts). A bivariate Dirichlet distribution for (X, Y) has pdf

$$f(x,y) = Cx^{a-1}y^{b-1}(1-x-y)^{c-1}, 0 < x < 1, 0 < y < 1, 0 < y < 1-x < 1,$$

where $a > 0, b > 0$, and $c > 0$ are constants.

- (a) Show that $C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$.
- (b) Find the conditional distribution of $Y|X = x$.
- (c) Find the conditional mean $E(Y|X)$

Problem 5 (20 pts). Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential (θ), and $Y_1, \ldots, Y_m$ are exponential (μ) where θ and μ are the scale parameters.

- (a) Find the likelihood ratio test of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.
- (b) Show that the test in part (a) can be used on the statistic

$$T = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i}.$$

- (c) Find the distribution of T when H_0 is true and then show how to get the rejection region of a test of size α.

Problem 6 (10pts). Let $X_1, \ldots, X_n$ be *iid* Poisson (λ). Find a UMP size α hypothesis test of $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda < \lambda_0$.

Problem 7 (20 pts). Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, i = 1, \ldots, n$$

where $x_1, \ldots, x_n$ are fixed constants, and $\epsilon_1, \ldots, \epsilon_n$ are *iid* $N(0, \sigma^2)$, σ^2 unknown.

- (a) Find a two-dimensional sufficient statistic for (β, σ^2).
- (b) Find the MLE of β, and show that it is an unbiased estimator of β.
- (c) Find the distribution of the MLE of β.
- (d) Give a 95% confidence interval for β.