In-Class Statistics Masters and Ph.D. Qualifying Exam

January 2020

Instructions: *The exam has 7 multi-part problems. All of the problems will be graded. Write your code words on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. (16pts) Consider the random variable

$$Y = X_1 + I \cdot X_2$$

where X_1, X_2 and I are mutually independent with X_i distributed as Poisson with parameter λ_i and

$$I = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } (1-p). \end{cases}$$

(a) Find the moment generating function of Y.

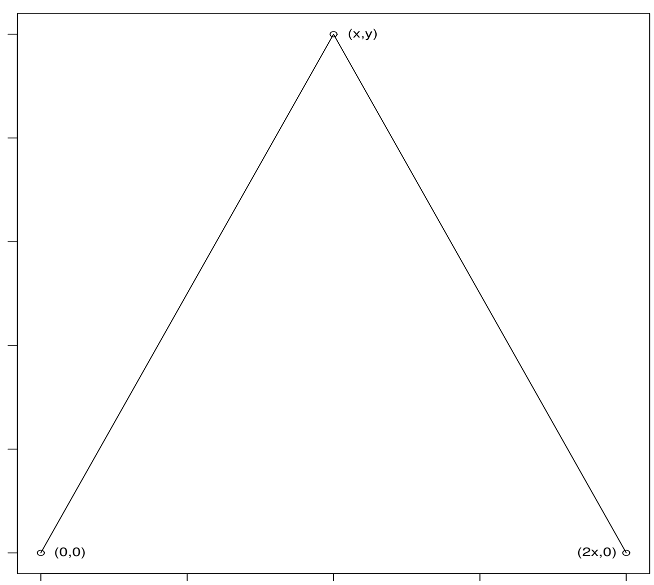
(b) Find the probability mass function of Y.

Problem 2 . (10pts) Let X and Y have a joint probability density function as follows:

$$f(x,y) = \begin{cases} 1 - \alpha(1-2x)(1-2y), & 0 < x < 1, 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

where $-1 \le \alpha \le 1$.
Suppose that an isoceles triangle is formed as in Figure 1, where (X, Y) has the above joint distribution.

Figure 1: Isoceles triangle for problem 2



Find the value of α which maximizes the expected area of the triangle.

Problem 3. (10pts) Let $X_1, X_2, \cdots, X_n$ be a random sample from the standard normal distribution $N(0,1)$. Let

$$\bar{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i, \quad \bar{X}_{n-k} = \frac{1}{n-k}\sum_{j=k+1}^{n} X_j.$$

Derive the distribution of

$$Y = k\bar{X}_k^2 + (n-k)\bar{X}_{n-k}^2.$$

Problem 4. (14pts) Let $(X_i, Y_i), i = 1, 2, \cdots, n$, be a collectrion of independent identically distributed random pairs with

$$E(X_i) = 25, E(Y_i) = 20$$

$$\text{var}(X_i) = 64, \text{var}(Y_i) = 100$$

$$\text{cov}(X_i, Y_i) = 32.$$

Let

$$S = \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{j=1}^{n} Y_j.$$

For large n, approximate $P(S \leq a)$, where a is a constant.

Problem 5. (20pts) Let $X_1, \ldots, X_n$ be a random sample of size n from the normal $(\theta, 1)$ distribution.

(a) Find the UMVUE of $\tau(\theta) = \theta^2$.

(b) Compute the Cramer-Rao lower bound for $\tau(\theta)$.

Problem 6. (15pts) Let $\bar{X}$ be the mean of a random sample of size n from some distribution having a finite mean μ and variance 10. Find n so that the interval $\left(\bar{X} - \frac{1}{2}, \bar{X} + \frac{1}{2}\right)$ contains μ with probability 0.9544.
Hint: if Φ is the standard normal cumulative distribution function, then $\Phi(2.0) = 0.9772$.

Problem 7. (15pts) Let $X_1, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)} & \text{for} x \geq \theta \\ 0 & \text{for } x < \theta. \end{cases}$$

Determine the generalized likelihood ratio statistic for testing $H_0 : \theta \leq \theta_0$ against $H_\alpha : \theta > \theta_0$.
Hint: You need not explicitly state the rejection region nor determine the distribution of the test statistic. Just find the test statistic itself.