In-Class Statistics Masters and Ph.D. Qualifying Exam

Spring 2021

Instructions: *The exam has 6 multi-part problems. The total score is 60. All of the problems will be graded. Write your code words on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. Let A be the set $\{1, 2, \ldots, n\}$. Let B and C each be random subsets of A, where all 2^n possible subsets are equally likely. For any set E, let $|E|$ be the number of elements of E.

(a) [4 points]. What is the probability that $B \subset C$?

(b) [3 points]. Find the expected value of $|B \cup C|$. (Hint: let $X_i = 1$ if $i \in B \cup C$ and consider the sum of the X_is.)

(c) [3 points]. Find the variance of $|B \cup C|$.

Problem 2 [5 points]. Suppose there are five coins. For coin i, $i = 1, \ldots, 5$, the probability of Heads is $i/10$. One coin is picked at random and flipped. The result is Heads. What is the probability that it was the third coin?

Problem 3. A person arrives at a certain bus stop at a time that is uniform between 9:00pm and 9:10pm. The last bus of the day arrives at the bus stop at a time that is X minutes after 9:00pm where X has density

$$f(x) = \begin{cases} \frac{1}{2}e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Assume that the times of arrival of the person and the bus are independent.

(a) [5 points]. Let the time that the person arrives be Y. Find the density of $X - Y$.

(b) [5 points]. If the person arrives at the bus stop and waits for more than 5 minutes without seeing the bus, the person calls a taxi (this could happen either because the person missed the bus by arriving late, or because the bus is late). Find the probability that the person calls a taxi.

Problem 4. Let $\mathbf{X} = \{X_1, \ldots, X_n\}$ be a random sample from a distribution with pdf given by $f(x|\theta) = \theta^{-c} c x^{c-1} e^{-(x/\theta)^c} I(x > 0)$, where $c > 0$ is unknown.

(a) [3 points] Find the MLE for θ.

(b) [3 points] Find the UMVUE for θ.

(c) [4 points] Find the uniformly most powerful test of size α for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ where θ_0 is a positive constant.

Problem 5. Let $\mathbf{X} = \{X_1, \ldots, X_n\}$ be a random sample from a distribution with pdf given by $f(x|\theta) = \theta^{-1}x^{(1-\theta)/\theta}I(0 \leq x \leq 1)$, where $\theta > 0$.

(a) [3 points] Show that $T(\mathbf{X}) = -2\sum_{i=1}^{n} \log X_i$ is a minimal sufficient statistic for θ.

(b) [3 points] Find the MLE of θ.

(c) [3 points] Show if the MLE is unbiased or not.

(d) [3 points] Find $I(\theta)$, Fisher's information for θ. Is the MLE the most efficient estimator?

(e) [3 points] Find a two-sided 95% confidence interval for θ based on T.

Problem 6. Let X_1 and X_2 be independent Poisson random variables with mean $\lambda > 0$. Based on these $n = 2$ observations, consider testing $H_0 : \lambda \leq 1$ versus $H_1 : \lambda > 1$.

(a) [3 points] Let $\phi_1 = \phi_1(X_1, X_2)$ denote the test that rejects H_0 if and only if $X_1 \geq 2$. That is,

$$\phi_1(X_1, X_2) = \begin{cases} 1 & X_1 \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the size of ϕ_1.

(b) [3 points] Let $\phi_2(t) = E[\phi_1(X_1, X_2)|X_1 + X_2 = t]$ be another test for the hypothesis. Derive a simple formula for $\phi_2(t)$.

(c) [4 points] Compare the power functions of ϕ_1 and ϕ_2.