In-Class Statistics Masters and Ph.D. Qualifying Exam

August, 2021

Instructions: *The exam has 5 multi-part problems. All of the problems will be graded. Write your ID number on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. (5 points per subproblem) A box contains 4 black balls, 5 red balls, and 6 green balls.

(a) Randomly draw two balls without replacement, what is the probability that the two balls have same color?

(b) Randomly draw three balls without replacement, what is the probability that the three balls have different colors (i.e., all three colors occur)?

(c) Randomly draw continuously with replacement, how many draws needed, on average, to see all three colors?

Problem 2. (5 points per subproblem) Let $X_i, i = 1, 2, \cdots$, be iid with density function

$$f(x) = \begin{cases} 2(1-x), & \text{for } 0 < x < 1 \\ 0 & \text{else.} \end{cases}$$

and let $Y_n = \min(X_1, X_2, \cdots, X_n)$.

(a) Find the cdf of Y_n, $F_n(y) = P(Y_n \le y)$.

(b) Find the constant c, such that $Y_n \xrightarrow{P} c$, and verify the convergence.

(c) Show that $D_n = nY_n$ converges in distribution and find the limiting distribution.

Problem 3. (15 points) X and Y are independent, standard normal random variables. Determine the conditional distribution of X given that $X - Y = v$

Problem 4. (5 points per subproblem) Let $X_1, \ldots, X_n$ be a random sample from $f(x|\theta)$ where

$$f(x|\theta) = \sqrt{\frac{2}{\pi}} \frac{x^2}{\theta^3} \exp\left[\frac{-x^2}{2\theta^2}\right] I(x > 0)$$

for $\theta > 0$. For this distribution, $E[X] = 2\theta\sqrt{2/\pi}$ and $Var(X) = \theta^2(3\pi - 8)/\pi$.

(a) Find a minimal sufficient statistic for θ.

(b) Find an M.O.M. estimate for θ^2.

(c) Find a Maximum Likelihood estimate for θ^2.

(d) Find the Fisher information for $\tau = \theta^2$ in the sample of n observations.

(e) Does the M.L.E. achieve the Cramér-Rao Lower Bound? Justify your answer.

(f) Find the mean squared error of the M.L.E. for θ^2.

(g) Find an approximate 95% interval for θ based on the M.L.E.

(h) What is the M.L.E. for θ? Is this M.L.E. unbiased for θ? Justify your answer.

Problem 5. (5 points per subproblem) Let X_1, X_2, X_3 be i.i.d. discrete random variables with $X_i \in \{0, 1, 2\}$ for $i = 1, 2, 3$ with pmf

$$P(X_i = x) = \frac{\lambda^x}{x!(1 + \lambda + \lambda^2/2)} I(x \in \{0, 1, 2\})$$

for $\lambda > 0$. Suppose that $X_1 = 1, X_2 = 2, X_3 = 2$ is the observed data.

(a) Find the M.L.E. for λ.

(b) Consider testing the null hypothesis $H_0 : \lambda \leq 2$ versus $H_1 : \lambda > 2$ Describe the form of a UMP level α test.

(c) For the context in problem (b), what is the one-sided p-value for the observed data? Give your answer as an exact fraction. Is there sufficient evidence to reject the null hypothesis at the $\alpha = .05$ level?