In-Class Statistics Masters and Ph.D. Qualifying Exam

January, 2022

Instructions: *The exam has 5 multi-part problems. All of the problems will be graded. Write your ID number on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. (15 pts) Let X and Y be independent unif(0,1) random variables. Recall that for this distribution, $Var(X) = 1/12$.

(a) Find the joint density of $U = X - Y$ and $V = X$. Be sure to specify the density over the entire real line.

(b) Find the marginal distribution of U.

(c) Find $Cov(U, V)$

Problem 2. (15 pts) Let (X_1, X_2, X_3) be multinomial$(3, .5, .2, .3)$. Recall that the multinomial distribution has pmf

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_m = x_m) = \frac{n!}{n_1!n_2!\ldots n_m!} p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$$

(a) Find $E[X_1]$.

(b) Find $E[X_1^3]$. You can either compute this directly or use the moment generating function for a binomial with parameters n and p:

$$((1-p) + pe^t)^n$$

(c) Find the probability that X_1 and X_2 are both greater than 0.

Problem 3. (20 pts) Let $P(Y_i = j) = 1/4$ for $j = 1, 2, 3, 4$ and $i = 1, 2$, and otherwise $P(Y_i = j) = 0$. Similarly, let $P(Z_i = j) = 1/6$ for $j = 1, 2, 3, 4, 5, 6$ and $i = 1, 2$, and otherwise $P(Z_i = j) = 0$. Let $\theta = 1$

with probability p and $\theta = 0$ with probability $1-p$. Assume that Y_1, Y_2, Z_1, Z_2 and θ are independent. Now define X to be $Y_1 + Y_2$ if $\theta = 1$ and $X = Z_1 + Z_2$ if $\theta = 0$.

(a) Find $E[X]$.

(b) Find $Var(X)$.

(c) Find the maximum likelihood estimate for θ if $X = 6$.

(d) Find $P(\theta = 1 | X = 6)$ if $p = 1/3$.

Problem 4. (35 pts) Let $X_1, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x|\lambda) = \lambda e^{-\lambda x} \text{ for } \lambda > 0.$$

(a) Find the method of moment estimator for λ.

(b) Find the MLE for λ.

(c) Find a level $1-\alpha$ confidence interval for λ.

(d) Find the UMVUE for $\tau(\lambda) = 1/\lambda^2$.

For the following problems, assume that the prior distribution for λ is Gamma(5,10). Recall that a Gamma distribution with parameters α and β is defined as follows:

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

for $0 \le x < \infty, \alpha, \beta > 0$.

(e) What is the posterior distribution of λ i.e. $\pi(\lambda|\mathbf{x})$. (Recall that $\pi(\lambda|x) \propto f(x|\lambda)\pi(\lambda)$ where $\pi(\lambda)$ is the prior distribution with Gamma(5,10).

(f) What is the posterior mean of $\theta = e^{-\lambda}$ i.e. $E(e^{-\lambda}|\mathbf{x})$.

(g) Find a consistent estimator of θ and show that it is consistent.

Problem 5. (15 pts) Let $X_1, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x) = \theta x^{\theta-1} \text{ for } 0 < x < 1 \text{ and } \theta > 0$$

(a) Use the Neyman-Pearson Lemma to find the most powerful test for testing

$$H_0 : \theta = \theta_0 \text{ versus } H_\alpha : \theta = \theta_1$$

at level α, assume that $\theta_1 > \theta_0$

(b) Find a level α uniformly most powerful test for testing

$$H_0 : \theta = \theta_0 \text{ versus } H_\alpha : \theta > \theta_0$$

at level α.

(c) For a sample size of $n = 1$ and $\alpha = 0.05$, explicitly compute the power function of the test in part (b).