In-Class Statistics Masters and Ph.D. Qualifying Exam

August, 2022

Instructions: *The exam has 6 multi-part problems. All of the problems will be graded. Write your ID number on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. (10 pts) Let $X_1, X_2, \ldots$ be a sequence of independent random variables, each uniformly distributed on the interval of $(0, 2)$. Let

$$Y = \sqrt{X_1} + \sqrt{X_1 X_2} + \sqrt{X_1 X_2 X_3} + \cdots,$$

find the mean of the random variable Y.

Problem 2. (10 pts) 1. Suppose $X|p$ has a Binomial distribution with parameters n and p, and assume p has a beta(1,1) distribution distribution. Recall that a beta(α, β) pdf is defined as follows:

$$f(x|\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

a. Find the distribution of X.

b. Find $E(X^2 + 2X)$.

Problem 3. (15 pts) A box contains 4 coins and each has a different probability of showing heads. Let $p_1, \ldots, p_4$ denote the probability of heads in each coin. Suppose that $p_1 = 0; p_2 = 1/4; p_3 = 1/2; p_4 = 1$. Let H denote "heads is obtained" and let C_i denote the event that coin i is selected.

(a) Select a coin at random and toss it. Suppose a head is obtained. What is the probability that coin i was selected ($i = 1, \ldots, 4$)? In other words, find $P(C_i|H); i = 1, \ldots, 4$.

(b) Toss the coin again. What is the probability of another head? In other words find $P(H_2|H_1)$ where H_j = "heads on toss j".

(c) Now suppose that the experiments was carried as follows: select a coin at random and toss until a head is obtained. Find $P(C_i|T_4)$ where T_4 = "first head is obtained on toss 4".

Problem 4. (15 pts)

Let X_1 and X_2 be a random sample with the following *pdf*:

$$f(x) = \begin{cases} 2.5x^4 & \text{for } -1 \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the *cdf* and the *pdf* for $Y = X_1^2$.

(b) Let $X_{(2)} = \max(X_1, X_2)$. Find the *pdf* for $X_{(2)}$.

(c) What is the correlation between X_1 and $X_1 + X_2$?

Problem 5. (25 pts)

Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from the discrete pmf

$$P(X = x) = \begin{cases} 1 - (2/3)\theta & x = 1 \\ (1/3)\theta & x \in \{2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

(a) For which values of θ is this a probability distribution?

(b) Write the log-likelihood function

(c) Find the maximum likelihood estimate for θ when $n = 6$ and the data are

$$3, 1, 1, 2, 1, 2$$

(d) Is the maximum likelihood estimator for θ unbiased for general n? Justify your answer.

(e) Find the Fisher information for one observation from this distribution.

Problem 6. (25 pts)

Let $X_1, \ldots, X_n$ be an i.i.d. sample from the distribution with pdf

$$f(x|\tau) = e^{-(x-\tau)} I(x \geq \tau)$$

Here $\tau \in (-\infty, \infty)$.

(a) Find a sufficient statistic for τ.

(b) Find the method of moments estimator for τ.

(c) Determine the form of a likelihood ratio test for testing $H_0 : \tau \leq 1$ versus $H_1 : \tau > 1$.

(d) Find the UMP most powerful α level test for testing $H_0 : \tau = 0$ versus $H_1 : \tau = 1$.

(e) Find a $(1 - \alpha) \times 100\%$ confidence interval for τ.