Statistics comprehensive exam. January 2023

Instructions: *The exam has 5 equally weighted problems. All parts of all problems will be graded. Write your code words on each of your answer sheets. Do not put your name or UNM ID on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. Let $X_1, \dots, X_n$ be i.i.d. $N(\theta, 1)$.

(a) Suppose the parameter space is restricted to $\theta \in \Theta = \{0, 1\}$. What is the MLE for θ?

(b) In the same setting as (a), show whether the MLE, $\widehat{\theta}$, is biased or unbiased.

(c) Also in the same setting as (a), what is the distribution of $\widehat{\theta}$?

(d) Now suppose the parameter space is $\Theta = \{0, 1, 2, \dots\}$ (i.e., the nonnegative integers). Find the MLE for θ.

Problem 2. Consider the linear model

$$Y = X\beta + Z\gamma + e, \qquad E(e) = 0, \qquad \text{Cov}(e) = V.$$

The matrix

$$A \equiv X[X'V^{-1}X]^{-}X'V^{-1}$$

is the oblique projection operator onto $C(X)$ along $C[V^{-1}X]^{\perp}$, i.e., if $v \in C(X)$, $Av = v$, and if $v \in C[V^{-1}X]^{\perp}$, $Av = 0$. Note that

$$(I - A)'V^{-1}(I - A) = (I - A)'V^{-1} = V^{-1}(I - A).$$

Let $\mathcal{A}_{X,Z}$ be the oblique projection operator onto $C(X, Z)$ along $C[V^{-1}(X, Z)]^{\perp}$. Assume that all matrix inverses exist.

(a) Show that

$$\mathcal{A}_{X,Z} = A + (I - A)Z[Z'(I - A)'V^{-1}(I - A)Z]^{-1}Z'(I - A)'V^{-1}.$$

(b) Show that

$$\hat{\gamma} = [Z'(I - A)'V^{-1}(I - A)Z]^{-1}Z'(I - A)'V^{-1}(I - A)Y$$
$$X\hat{\beta} = A(Y - Z\hat{\gamma})$$

provide generalized least squares estimates (BLUEs) for the linear model.

Problem 3.

(a) State and prove the Neyman-Pearson Lemma for testing a simple null hypothesis vs a simple alternative.

(b) Let $X_1, X_2, ...$ be independent with X_n taking the values $\sqrt{n}$, 0, and $-\sqrt{n}$, each with probability $^1/_3$. Find the asymptotic distribution of

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Problem 4. Let $y_1, ..., y_n \overset{\text{iid}}{\sim} \text{Bern}(\theta)$ and let

$$\theta = \frac{e^\delta}{1+e^\delta}.$$

Raiffa and Schlaifer (1961) argue that if one is ignorant about the parameters in a model, one should be equally ignorant about any transformation of those parameters.

(a) Consider the flat prior on δ, $p_a(\delta) \propto 1$, the improper continuous uniform over the Real line. Use change of variables to identify the induced prior this creates on θ, $p_a(\theta)$.

(b) Consider the flat prior on θ, $p_b(\theta) = 1 \times I_{(0,1)}(\theta)$, the continuous uniform on the unit interval. Use change of variables to identify the induced prior this creates on δ, $p_b(\delta)$.

(c) Along with the Jeffreys prior on θ, $p_a(\theta)$ and $p_b(\theta)$ are the most common reference priors used with Binomial data. State or derive the Jeffreys prior for θ, $p_c(\theta)$.

(d) Consider the shapes of the three θ priors above. Which values of θ are best supported (i.e. have the highest prior density) under $p_a(\theta)$? Which values of θ are best supported under $p_b(\theta)$. Which values are best supported under $p_c(\theta)$?

(e) Respond to Raiffa and Schlaifer's argument. Can one be "equally ignorant" about both θ and δ? Is a flat prior an effective way to express ignorance?

Problem 5. Consider a linear model

$$Y = X\beta + e, \qquad \text{E}[e] = 0$$

with the (not necessarily estimable) linear constraint $\Lambda'\beta = d$.

(a) Characterize the reduced model associated with this constraint (hypothesis).

(b) Consider two solutions to the constraint, b_1 and b_2, so that $\Lambda' b_k = d$, $k = 1, 2$. Define appropriate least squares fitted values $\hat{Y}_k$.from the corresponding reduced models.

(c) Show that $\hat{Y}_1 = \hat{Y}_2$. Hints: Show that $(I - M_0)X(b_1 - b_2) = 0$. In what space does $(b_1 - b_2)$ lie?