In-Class Statistics Masters and Ph.D. Qualifying Exam

January, 2023

Instructions: *The exam has 7 multi-part problems. All of the problems will be graded. Write your ID number on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.*

Problem 1. (15 pts) The following experiment corresponds to randomly picking a die with either 4, 6, 8, or 10 sides and then rolling it once. Let Y be a discrete random variable with pmf

$$P(Y = y) = \begin{cases} 1/4 & y \in \{4, 6, 8, 10\} \\ 0 & \text{otherwise} \end{cases}$$

Let $X|Y = y$ be discrete uniform on $\{1, 2, \ldots, y\}$. In other words

$$P(X = x|Y = y) = \begin{cases} 1/y & x \in \{1, 2, \ldots, y\} \\ 0 & \text{otherwsie} \end{cases}$$

Note that if U is discrete uniform on $\{1, 2, 3, 4\}$, then $Y = 2U + 2$. For a discrete uniform U on $\{1, 2, \ldots, n\}$,

$$E[U] = (n + 1)/2 \quad Var(U) = (n^2 - 1)/12$$

(a) Find $E[X]$.

(b) Find $Var(X)$.

(c) Suppose the value $X = 3$ is observed but it is unknown which die was used to roll the 3. Assuming that each die was equally likely to be chosen, find the probability that the die had six sides given that $X = 3$.

Problem 2. (5 pts) For any events A and B, show that $P(A|A \cup B) \geq P(A|B)$. Hint: try calculating the left hand side by conditioning on whether B occurs.

Problem 3. (5 pts) A fair coin is flipped until the number of heads exceeds the number of tails by two. Let X be the number of flips. For example, if the sequence is HH, then $X = 2$. If the sequence is $THTTHHHH$, then $X = 8$.

Find $P(X = 10)$.

Problem 4. (20 pts) Let X have density

$$f_X(x) = e^{-(x-1)} I(x > 1)$$

(a) Write the CDF for X. Be sure to specify the CDF over the entire real line.

(b) Find $P(X > 3|X > 2)$

(c) Does this distribution have the memoryless property? Justify your answer.

(d) Find the density of $Y = (1 + X)^{-1}$. Be sure to specify the density over the entire real line.

Problem 5. (5 pts)

Let X and Y be independent with densities

$$f_X(x) = 1\ I(1 < x < 2), \quad f_Y(y) = \frac{1}{y^2}\ I(y > 1)$$

Find the density of $U = X+Y$. Specify the density of U over the entire real line.

Problem 6. (20 pts) Let $X_1, X_2, \cdots X_n$ be a random sample with densities given by the p.d.f. $f(x;\theta) = e^{-(x-\theta)}$ for $x > \theta$ and zero otherwise, where $-\infty < \theta < \infty$.

(a) Find and justify a sufficient and complete statistic for θ.

(b) Determine the UMVUE for $\tau(\theta) = \theta$.

Problem 7. (30 pts) Suppose observbations are form the following density:

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) (7 pts) Suppose that we wish to test

$$H_0 : \theta = 4 \text{ vs. } H_1 : \theta = 3.$$

Construct the Most Powerful (MP) level-0.05 test based on **one** observation $X|\theta \sim f(x|\theta)$.

(b) (7 pts) What is the power of the test in (a)?

(c) (8 pts) If we have $X_1, X_2, \ldots, X_n|\theta \overset{iid}{\sim} f(x|\theta)$, what is the distribution of $T(\boldsymbol{X}) = \sum_{i=1}^{n} \log(x_i)$?

(d) (8 pts) Based on this random sample of size n, find a level-α UMP critical region for testing $H_0 : \theta \le 3$ against $H_\alpha : \theta > 3$.