In-Class Statistics Masters and Ph.D. Qualifying Exam

August, 2023

Instructions: *The exam has 4 multi-part problems. All of the problems will be graded. Write your ID number on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained.* **Do not use calculators, cell phones, or other electronic devices.**

Problem 1. (5 points per subproblem) Players a, b, and c take turns flipping a coin with $P(H) = p$ and $P(T) = q = 1 - p$. When one player first gets heads, that player stops flipping and the remaining players continue flipping. For example, if the players start in the sequence a,b,c, then a possible sequence where b is the first player to get heads and c is the second player to get heads is is a, b, c, a, b, c, a, c, a, c, a, a, a. Here b got heads on the second try for that player, c got heads on the 4th try for that player, and a got heads on the 7th try.

(a) Find the probability that player a gets heads first, then player b, then c (but not necessarily on their initial tries).

(b) Assuming that the initial sequence of three players is random (such as a,b,c or b,c,a, or c,b,a etc.), what is the probability that the initial sequence was a,b,c if player a got heads before players b or c?

(c) What is the expected number of flips until all three players get heads. In the above example (before part (a)), the total number of flips was 13.

Problem 2. (5 points per subproblem)

Let X_1, X_2, X_3 follow a multinomial distribution,

$$P(X_1 = n_1, X_2 = n_2, X_3 = n_3) = \frac{n!}{n_1!n_2!n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3}$$

where $n_i + n_2 + n_3 = n$ is the number of trials and for each trial, p_i is the probability that that trial is in category i. Here X_i is a count of the number of times category i is observed.

(a) Derive the formula for $Cov(X_1, X_2)$.

(b) What is the marginal distribution for X_1?

Problem 3. (5 points per subproblem) Let $X_1, X_2, \dots, X_n$ be i.i.d. from a sample with density

$$f(x|\tau) = e^{-(x-\tau)}\ I(x > \tau)$$

(a) Consider the hypothesis test where

$$H_0 : \tau = 0, \quad H_1 : \tau > 0$$

What is the UMP $\alpha = 0.05$ level test?

(b) Find the likelihood ratio statistic for this problem

Problem 4. (5 points per subproblem) Let X_i, $i = 1, \dots, n$ be independent and exponentially distributed with $E[X_i] = i\theta$

$$f(x_i|\theta) = \frac{1}{i\theta} e^{-x_i/(i\theta)}\ I(x > 0)$$

(a) Find the maximum likelihood estimate of θ.

(b) Determine whether the maximum likelihood estimate is unbiased and justify your answer.

(c) Find a sufficient statistic for θ.