Statistics comprehensive exam. January 2024

Instructions: The exam has 6 problems often with multiple parts. **Write your code words on each of your answer sheets**. *Do not put your name or UNM ID on any of the sheets.* **Be clear, concise, and complete. All solutions should be rigorously explained.**

Problem 1. Consider testing the null hypothesis $Y \sim \text{Bin}(5, 0.2)$ versus the alternative $Y \sim \text{Bin}(5, 0.4)$. Use the Neyman-Pearson Lemma to find the most powerful size 0.05 test.

Here is some R output that you might find useful.

```
> t
[1] 0 1 2 3 4 5
> dbinom(t,5,0.2)
[1] 0.32768 0.40960 0.20480 0.05120 0.00640 0.00032
> dbinom(t,5,0.4)
[1] 0.07776 0.25920 0.34560 0.23040 0.07680 0.01024
```

Why is this also the UMP test for $H_0 : p \leq 0.2$ versus $H_1 : p > 0.2$, when $Y \sim \text{Bin}(5, p)$?

Problem 2. Consider a linear model

$$Y = X\beta + e, \quad \text{E}[e] = 0, \quad \text{Cov}[e] = \sigma^2 I.$$

Let M be the perpendicular projection operator onto $C(X)$. Consider the parameter $\lambda'\beta$ where λ is known. For known vectors a and ρ, suppose $a'Y$ and $\rho'Y$ are both unbiased estimates of $\lambda'\beta$.

(a) Show that $\lambda'\beta$ is estimable.

(b) Show that $a'X = \rho'X$.

(c) Show that $\text{Var}[a'Y] = \text{Var}[a'Y - \rho'MY] + \text{Var}[\rho'MY]$.

(d) State and prove the Gauss-Markov Theorem.

Problem 3. Let $W_1, ..., W_n \overset{\text{iid}}{\sim} \text{Bern}(p)$, and let $\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} W_i$.

(a) Use the Central Limit Theorem to find the limiting distribution for $\sqrt{n}(\hat{p}_n - p)$.

(b) Use the Delta Method to find the limiting distribution for $\sqrt{n}[g(\hat{p}_n) - g(p)]$, where $g(u) = \ln\left(\frac{u}{1-u}\right)$, the log-odds function.

Problem 4. Let $y_1, ..., y_n$ be a random sample modeled by a Pois (λ) distribution.

(a) Consider the case where λ is unknown. Find the score function for λ.

(b) Find the Fisher information for λ.

(c) Find the Jeffreys prior for λ.

(d) The Jeffreys prior is always a flat prior on some parameterization of the model it comes from. In the case where λ is unknown, find the parameterization for the model that gives rise to a flat Jeffreys prior over $\mathbb{R}$.

Problem 5. Consider a linear model

$$Y = X\beta + e, \qquad \mathrm{E}[e] = 0$$

with the (not necessarily estimable) linear constraint $\Lambda'\beta = d$.

(a) Characterize the reduced model associated with this constraint (the hypothesis).

(b) Consider two solutions to the constraint, b_1 and b_2, so that $\Lambda' b_k = d$, $k = 1, 2$. Give appropriate least squares fitted values $\hat{Y}_k$ from the models $Y = X_0\gamma + Xb_k + e$ where $X_0 = XU$ with $C(U) = C(\Lambda)^{\perp}$.

(c) Show that $\hat{Y}_1 = \hat{Y}_2$. Hint: After finding $\hat{Y}_k$, show that $(I - M_0)X(b_1 - b_2) = 0$ where M_0 is the perpendicular projection operator onto $C(X_0)$.

Problem 6. Let $X_1, X_2, \ldots$ be independent with X_n taking the values $\pm\sqrt{n}$ each with probability 1/2. Show that $\overline{X}_n$ converges in distribution to a $N(0, 1/2)$.