

## In-Class Statistics Masters and Ph.D. Qualifying Exam

August, 2023

**Instructions:** *The exam has two multi-part problems. All of the problems will be graded. Write your ID number on each of your answer sheets. Do not put your name on any of the sheets. Be clear, concise, and complete. All solutions should be rigorously explained. Do not use calculators, cell phones, or other electronic devices.*

**Problem 1.** (5 points per subproblem) Consider a bowl with 5 blue marbles numbered  $1, \dots, 5$  and 7 green marbles numbered  $1, \dots, 7$ . For (a)–(c), Two marbles are selected at random without replacement.

- (a) What is the probability that they have the same number?
- (b) What is the probability that they have the same color or the same number?
- (c) What is the probability that the first marble is green given that the second marble is blue?
- (d) Assume that  $k$  yellow marbles are added to the bowl, where  $k$  is unknown. Assuming that you can select  $n$  marbles with replacement, explain how you could estimate  $k$  when sampling with replacement.
- (e) Again, assuming that  $k$  yellow marbles are added to the bowl where  $k$  is unknown, if you wanted to estimate  $k$  from a sample of  $n = 5$  observations, would you rather sample with replacement or without replacement to get a more precise estimate of  $k$ ? Explain your reasoning.

**Problem 2.** (5 points per subproblem)

Let  $X_1, \dots, X_n$  be an i.i.d. sample from density

$$f(x|\theta) = \frac{x}{\theta} e^{-x^2/\theta} I(x > 0)$$

where  $\theta > 0$ .

- (a) Let  $Y = X_1^2$ . Find the density for  $Y$ .
- (b) Find a Method of Moments estimator of  $\theta$ .
- (c) Find the Maximum Likelihood estimator of  $\theta$ .
- (d) Find the Maximum Likelihood estimator of  $\tau = \theta^2$ .
- (e) Find the Cramér-Rao lower bound for an unbiased estimator of  $\theta$ .
- (f) Does the Method of Moments estimator achieve the Cramér-Rao lower bound?
- (g) For the purpose of getting a confidence interval, show that  $\sum_{i=1}^n X_i^2/\theta^2$  is a pivotal quantity.
- (h) Show, in as much detail as possible, how to get a 95% confidence interval for  $\theta$  using the pivotal quantity in (g).
- (i) Consider the likelihood ratio test of  $H_0 : \theta = 1$  vs  $H_1 : \theta \neq 1$ . Describe the likelihood ratio test in as much detail as possible.