Geometry/Topology Qualifying Exam, August, 2024
Department of Mathematics & Statistics
University of New Mexico

Instructions:

- You have 3 hours to complete the exam. Please try all of the 10 problems on the exam. You can work on any of the parts of the given problems assuming all previous parts even if you skip some of them. Clear and concise answers with good justification will improve your score.
- Use only one side of the pages. Start each problem on a new page.
- Number the pages so that the problems appear in the order in which they appear on the exam.
- Put only your code word (not your banner ID number) on each page.

(1) Let X be a Hausdorff topological space and f and g be two continuous maps $f: X \to X$ and $g: X \to X$. Show that the coincidence set of f and g is closed in X, i.e., the set $C = \{x \in X \mid f(x) = g(x)\}$ is a closed set in X.

(2) Let $\pi : E \to M$ be a smooth vector bundle over a smooth manifold M. Let $X \in \Gamma(TM)$ be a fixed smooth vector field on M and $\mathcal{F}_X : \Gamma(E) \to \Gamma(E)$ be an $\mathbb{R}$-linear map which satisfies the Leibniz rule

$$\mathcal{F}_X(f\sigma) \;=\; df(X)\,\sigma \;+\; f\mathcal{F}(\sigma), \qquad f \in \mathcal{S}(M),\ \sigma \in \Gamma(E),$$

where $\mathcal{S}(M)$ denotes the space of smooth functions on M. Show that $\mathcal{F}_X$ is a local operator, i.e., if two sections σ, $\sigma' \in \Gamma(E)$ coincide on an open set U, $\sigma|_U = \sigma'|_U$, then $\mathcal{F}_X(\sigma)|_U = \mathcal{F}_X(\sigma')|_U$. Note: we do not assume that $\mathcal{F}_X$ is an $\mathcal{S}(M)$-linear map.

(3) a) Suppose M is $m-$dimensional smooth compact manifold (without boundary) and $[\omega_0] \in H^m_{dR}(M)$. Show that the integral $\int_M \omega$ is constant on the cohomology class $[\omega_0]$ determined by ω_0.

b) Suppose M is a smooth three dimensional compact manifold (without boundary) and $[\omega] \in H^1(M)$. Show that if γ is the positively oriented boundary of some hypersurface S in M, $\partial S = \gamma$, then

$$\int_\gamma \omega = 0.$$

(4) a) Show that the following formula defines a smooth action of the integer numbers (as a discrete Lie group) $\mathbb{Z}$ on the real numbers $\mathbb{R}$ (as additive group):

$$(n, x) \mapsto 3^n x$$

b) Show that the action is not proper. Note: make sure to state and verify the conditions for a smooth action.

(5) Let X and Y be topological spaces and π be the projection $\pi : X \times Y \to X$.
   a) Give an example where π is not a closed map.
   b) Prove that if Y is compact, then the projection $\pi : X \times Y \to X$ is a closed map.

(6) Let D be the closed unit disc in the two-dimensional plane. Let S be boundary of D. Show that if $f : D \to D$ is a homeomorphism then f maps S homeomorphically to itself. Hint: consider the induced map between the fundamental groups of $S \setminus x_0$ and $S \setminus f(x_0)$ for suitable x_0.

(7) Let M be a 3-dimensional smooth manifold, and $\theta \in \Omega^1(M)$ a smooth differential form such that $\theta \wedge d\theta$ is a volume form.
   a) Prove that there does not exist a 2-dimensional submanifold S of M such that $T_pS = \ker \theta_p$ for every $p \in S$.
   b) Show that there is a unique smooth vector field R such that $\theta(R) = 1$ and $\iota_R d\theta = 0$.

(8) Show that the dimension of the orthonormal Lie group $O(n)$ is $\frac{n^2-n}{2}$.

(9) Show that every Lie group G is an orientable manifold.

(10) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^3 - 6xy + y^2$. Find all values $c \in \mathbb{R}$ for which the level set $f^{-1}(c)$ is a regular submanifold of $\mathbb{R}^2$.