Linear and Rational Equations

Linear Equations in One Variable

1. Solve (a)
$$2x - 3(2x+2) = 1 - 5(4x+3)$$
 (b) $0.6x - 1.3 = 0.2(5x-8)$

(c)
$$\frac{2}{3}x - \frac{5}{6}x - 3 = \frac{1}{2}x - 5$$

What's my 1st move?

$$I^{st} \text{ move: Distribute} \\ 2x - 6x - 6 = 1 - 20x - 15 \\ -4x - 6 = -20x - 14 \\ 16x = -8 \\ x = -\frac{1}{2} \\ \text{Solution: } \{-\frac{1}{2}\} \end{cases} \qquad \begin{bmatrix} 1^{st} \text{ move: M! both sides by 10} \\ 10[0.6x - 1.3] = 10[0.2(5x - 8)] \\ 6x - 13 = 2(5x - 8) \\ 6x - 13 = 10x - 16 \\ -4x = -3 \\ x = \frac{3}{4} \\ \text{Solution: } \{\frac{3}{4}\} \end{bmatrix} \qquad \begin{bmatrix} 1^{st} \text{ move: M! both sides by the LCD, 6} \\ 6[\frac{2}{3}x - \frac{5}{6}x - 3] = 6[\frac{1}{2}x - 5] \\ 4x - 5x - 18 = 3x - 30 \\ -x = 3x - 12 \\ -4x = -12 \\ x = 3 \\ \text{Solution: } \{\frac{3}{4}\} \end{bmatrix}$$

Rational Equations

A rational equation is an equation in which a variable appears in the denominator of a fraction. We must restrict the values of the variable to avoid division by zero!

2. Solve (a)
$$\frac{9}{4} - \frac{1}{2x} = \frac{4}{x}$$
 restrictions: $x \neq 0$ (b) $\frac{1}{y-1} + \frac{y+1}{y^2+2y-3} = \frac{1}{y+3}$ restrictions: $y \neq 1, -3$

What's my move?

1st move: M! both sides by LCD,
$$4x$$
Step 1:F! denominators: $\frac{1}{y-1} + \frac{y}{(y+3)(y-1)} = \frac{1}{y+3}$ $4x \left[\frac{9}{4} - \frac{1}{2x}\right] = 4x \left[\frac{4}{x}\right]$ Step 1:F! denominators: $\frac{1}{y-1} + \frac{y}{(y+3)(y-1)} = \frac{1}{y+3}$ $9x - 2 = 16$ Step 2:M! both sides by the LCD: $9x = 18$ Step 3:Distribute and R! $y + 3 + y = y - 1$ $x = 2$ Solution: {2}Solve for y: $y = 4$

(c)
$$\frac{u-3}{u-6} = \frac{u+3}{u+2} - 1$$

Step 1: M! both sides by the LCD:
 $(u-6)(u+2)(\frac{u-3}{u-6}) = (\frac{u+3}{u+2} - 1)(u-6)(u+2)$
Step 2: Distribute and R! $(u+2)(u-3) = (u+3)(u-6) - (u-6)(u+2)$
Step 3: Solve for $u: u^2 - u - 6 = u^2 - 3u - 18 - (u^2 - 4u - 12)$
 $-u - 6 + 3u + 18 = -u^2 + 4u + 12$
 $u^2 - 2u = 0$
 $u(u-2) = 0 \Rightarrow u = 0, 2$
YT#2: Solve (a) $\frac{2}{x+1} - \frac{x}{x^2 - x - 2} = \frac{3}{x-2}$
(b) $6 + \frac{5}{y-2} = -\frac{2}{y-1}$

Absolute Value Equations

Example:

$$|x| = 4$$
 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6

If k > 0, then |u| = k is equivalent to u = k or u = -k.

3. Solve (a)
$$2|x-4|+1=7$$
(b) $|3x-1| = |x+5|$ (c) $|x+2| = -3$ Step 1:ISOLATE the absolute value:
 $|x-4| = 3$ Step 1:One abs value ISOLATED, so now...
 $|3x-1| = |x+5|$ SOLUTION?Step 2:INSIDE must be 3 or -3:
 $x-4=3$ or $x-4=-3$ Step 2:INSIDE of L must equal \pm INSIDE of R:
 $3x-1=x+5$ or $3x-1=-(x+5)$ Solution!
How can abs value =
something negative?Step 3:Solve for x:
 $x=7$ or $x=1$
Solution: {7, 1}Step 3:Solve for x:
 $x=3$ or $x=-1$ Solution: {3, -1}

Absolute Value Inequalities



4. Solve (a) $|2x + 5| + 2 \le 11$

(b) 5|2x + 3| - 1 > 9

Step 1: ISOLATE absolute value: SOLUTION? ISOLATE absolute value: |2x + 3| > 2 $|2x + 5| \le 9$ ANY value of x will make the statement Step 2: Quantity INSIDE is MORE than 2 from 0 – Quantity INSIDE is WITHIN 9 of 0 -TRUE because ABS Distance Q lies from 0 is MORE than 2: Distance Q lies from o is LESS than 9: VALUE is NEVER negative! 2x + 3 < -2 or 2x + 3 > +2 $-9 \le 2x + 5 \le 9$ Solve for *x*: Sol'n: $(-\infty, \infty)$ Solve for *x*: $-14 \le 2x \le 4$ $x < -\frac{5}{2}$ or $x > -\frac{1}{2}$ Step 3: $-7 \le x \le 2$ (d) |6-5x| < -4**Interval Notation: Interval Notation:** [-7, 2] SOLUTION? $\left(-\infty,-\frac{5}{2}\right)\cup\left(\frac{1}{2},\infty\right)$ Absolute value can NOT be negative, so (b) $\left| -\frac{2}{3}(x+4) - 2 \right| + 2 \le 6$ there is no solution! **YT#3** Solve (a) 2|5 - 2x| - 3 > 5

(c) $|2x + 1| \ge 0$

Sol'n: Ø

5. Concept Check

(a) Consider the statement |W| < 5. Write a statement in words that explains what it means, and then draw a graph and solve.



1) Draw a graph that represents this statement. 2) Write it as an equivalent absolute value statement.

(b) Consider the statement |x| > 4.

1) Write a statement in words that explains what it means. 2) Draw a graph. 3) Solve using inequality notation.

Quadratic Equations

Solve Quadratic Equations by Using the

What's my 1st move?

6. Solve (a) x(2x + 1) = 15*Can I set each factor = 0*? NO! Must have ZERO on right side!

> $2x^{2} + x = 15$ $2x^{2} + x - 15 = 0$ (2x -)(x -) = 0 (2x - 5)(x + 3) = 0 $2x - 5 = 0 \quad or \quad x + 3 = 0$ $x = \frac{5}{2} \quad or \quad x = -3$ $\{\frac{5}{2}, -3\}$

Zero Product Property: Square Root Property:

(b) $24x^2 = 8x$ *Can I divide both sides by x*? NO! x = 0 is a solution that we would LOSE You never want to divide both sides of an EQ by a variable quantity if you can help it!

(c)
$$2(x + 7)^2 - 16 = 0$$

What's my best move?
Add 16, divide by 2, then
SQUARE ROOT both sides

$$24x^{2} - 8x = 0$$

$$8x(3x - 1) = 0$$

$$8x = 0 \quad or \quad 3x - 1 = 0$$

$$x = 0 \quad or \quad x = \frac{1}{3}$$

$$\{0, \frac{1}{3}\}$$

 $2(x + 7)^{2} = 16$ (x + 7)² = 8 x + 7 = ±\sqrt{8} x + 7 = ±2\sqrt{2} x = -7 ± 2\sqrt{2} {7 ± 2\sqrt{2}}

YT#5: Solve (a) (x - 3)(2x + 1) = 3 (b) $2(x - 3)^2 - 40 = 0$

Quadratic Formula: For $ax^2 + bx + c = 0$ where $a \neq 0$, the solutions are



If mn = 0, then m = 0 or n = 0If $x^2 = k$, then $x = \pm \sqrt{k}$



Solve Equations for a Specified Variable 8. Solve $3x + 4a = \frac{2px+1}{2a}$ for *x*.

What's the overall idea here?

YT#7: Solve (a) $4a^2 - b^2 = 8$ for *a*. (b) $y = \frac{6x+1}{3x-5}$ for *x*. Overall goal: "Free" the x's so we can isolate the x terms, make the two x's "become 1, and solve for x.

Step 1: M! both sides by 2a	$2a(3x+4a) = 2a(\frac{2px+1}{2a})$
Step 2: Distribute	$6ax + 8a^2 = 2px + 1$
Step 3: Isolate <i>x</i> terms	$6ax - 2px = 1 - 8a^2$
Step 4: F! out the <i>x</i> :	$x(6a-2p) = 1 - 8a^2$
Step 5: Divide:	$x = \frac{1 - 8a^2}{6a - 2p}$

More Equations

9. Solve (a) $(u-3)^3 + 88 = 0$

Step 1: ISOLATE *u*-term: $(u - 3)^3 = -88$

Step 2: Cube root both sides: $u - 3 = \sqrt[3]{-88}$

Step 3: Add 3 to both sides and simplify radical:

$$u = 3 + \sqrt[3]{-8(11)}$$

$$u = 3 - 2\sqrt[3]{11}$$

9. Solve (a)
$$8x^3 + 4x^2 - 50x - 25 = 0$$

Group terms: $(8x^3 + 4x^2) - (50x + 25) = 0$ F! out GCF: $4x^2(2x + 1) - 25(2x + 1) = 0$ F! out GCF: $(2x + 1)[4x^2 - 25] = 0$ $2x + 1 = 0 \quad or \quad 4x^2 - 25 = 0$ $2x = -1 \quad or \quad 4x^2 = 25$ $x = -\frac{1}{2} \quad or \quad x^2 = \frac{25}{4}$ $x = \pm \frac{5}{2}$ $\{-\frac{1}{2}, \pm \frac{5}{2}\}$ **YT#8:** Solve (a) $-4\sqrt[3]{2x - 5} + 6 = 14$ (b) $98x^3 - 49x^2 - 8x + 4 = 0$ (c) $2x^3 - 128 = 0$ (Hint: divide both side: (b) $\sqrt[4]{2u+15} + 3 = 5$ Step 1: ISOLATE *u*-term: $\sqrt[4]{2u+15} = 2$ Step 2: Raise both sides to 4th power: 2u + 15 = 16Step 3: Subtract 15, divide by 2: 2u = 1 $u = \frac{1}{2}$

Must CHECK b/c we raised both sides to an EVEN power! It works.

(b) $27 - 8x^3 = 0$

Recognize as
$$A^3 - B^3$$
: $27 - 8x^3 = (3)^3 - (2x)^3$
 $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$: $A = 3, B = 2x$
 $27 - 8x^3 = (3 - 2x)(9 + 6x + 4x^2) = 0$
 $3x - 2 = 0 \quad \text{or} \quad 9 + 6x + 4x^2 = 0$
 $x = \frac{2}{3} \quad \text{or} \quad 9 + 6x + 4x^2 = 0$
 $x = -\frac{6\pm\sqrt{-108}}{8}$
 $x = -\frac{6\pm\sqrt{-36(3)}}{8}$
 $x = -\frac{6\pm6i\sqrt{3}}{8}$
 $x = -\frac{3\pm3i\sqrt{3}}{4}$
s by 2 1st!)

Solving Radical Equations

Step 1 Isolate the radical. If an equation has more than one radical, choose one of the radicals to isolate.

Step 2 Raise each side of the equation to a power that will "undo" the radical

Step 3 Solve the resulting equation. If the equation still has a radical, repeat steps 1 and 2.

Step 4 Check the potential solutions in the original equation and write the solution set.

In solving radical equations, extraneous solutions potentially arise when both sides of the equation are raised to an even power. Therefore, an equation with only <u>odd-indexed roots</u> will not have extraneous solutions.

10. Solve (a) $\sqrt{1 - 2x} - \sqrt{1 - x} = 2$

Step 1:	Isolate radical: $\sqrt{1-2x} = 2 + \sqrt{1-x}$
Step 2:	Square both sides: $(\sqrt{1-2x})^2 = (2+\sqrt{1-x})^2$ Square out & S! $1 - 2x = (2+\sqrt{1-x})(2+\sqrt{1-x})$
	$1 - 2x = 4 + 2\sqrt{1 - x} + 2\sqrt{1 - x} + 1 - x$ $-2x = 4 - x + 4\sqrt{1 - x}$
Step 3:	Isolate radical: $-4\sqrt{1-x} = 4 + x$
Step 4:	Square both sides: $\left[-4\sqrt{1-x}\right]^2 = (4+x)^2$
	$16(1-x) = 16 + 8x + x^2$
	$16 - 16x = 16 + 8x + x^2$
	$0 = x^2 + 24x$
	0 = x(x + 24)
	x = 0 or -24
Step 5:	CHECK!
	put in ORIGINAL EQ – $x = 0$ doesn't work! Solution: {-24}

(b)
$$x^{2/3} = 4$$
 (c) $(x+1)^{\frac{2}{5}} = 9$

Step 1:ISOLATE radical: doneStep 1:Step 2:Square root both sides: $\sqrt{x^{2/3}} = \pm \sqrt{4}$ Step 2: $x^{1/3} = \pm 2$ $x^{1/3} = \pm 2$ Step 3:Step 3:Cube both sides: $(x^{1/3})^3 = (\pm 2)^3$ Step 3: $x = \pm 8$ $x = \pm 8$

ISOLATE radical: already is isolated Raise both sides to 5th power: $[(x + 1)^{\frac{2}{5}}]^5 = 9^5$ $(x + 1)^2 = 9^5$ Square root both sides: $x + 1 = \pm \sqrt{9^{2}9^{2}9}$ $x + 1 = \pm 9(9)(3)$ $x = -1 \pm 243$ x = 242, -244CHECK answers because of squaring both sides! 242 does not work. Solution: x = -244.

YT#9: Solve (a) $\sqrt{k-2} = \sqrt{2k+3} - 2$ (b) $(2x-1)^{3/2} - 3 = 122$

YOUR TURN solutions start on the next page

YT solutions

Linear Equations and Rational Equations

YT#1: Solve
$$\frac{2}{3}(x-1)+2 = \frac{1}{6}x - \frac{1}{2}$$

$$\begin{bmatrix} 1^{n} \mod 1 + 2 = \frac{1}{6}x - \frac{1}{2} \\ 6 \left[\frac{2}{3}(x-1)+2\right] = \left[\frac{1}{6}x - \frac{1}{2}\right] 6 \\ 6 \left[\frac{2}{3}(x-1)+2\right] = \left[\frac{1}{6}x - \frac{1}{2}\right] 6 \\ 6 \left(\frac{2}{3}(x-1)+2\right) = 2 - 6 \cdot \frac{1}{6}x - 6 \cdot \frac{1}{2} \\ 4(x-1)+12 = x - 3 \\ 4x - 4 = x - 15 \\ 3x = -11 \\ x = -\frac{11}{3} \\ \\ Solution: \{-\frac{11}{3}\} \end{bmatrix}$$

YT#2: Solve (a) $\frac{2}{x+1} - \frac{x}{x^2-x-2} = \frac{3}{x-2}$
Step 1:
YT#2: Solve (a) $\frac{2}{x+1} - \frac{x}{x^2-x-2} = \frac{3}{x-2}$
Step 2:
YT#2: Solve (b) $6 + \frac{5}{y-2} = -\frac{2}{y-1}$
Step 3:
Distribute and R! $2(x-2) - x = 3(x+1)$
Solve for x : $2x - 4 - x = 3x + 3 - 2x = 7 \\ x = -\frac{7}{2} \\ Solution: \{-\frac{7}{2}\}$
Step 1:
M! both sides by the LCD: $(y-2)(y-1) \left[6 + \frac{5}{y-2}\right] = \left[-\frac{2}{y-1}\right](y-2)(y-1)$
Step 3:
Step 4:
M! both sides by the LCD: $(y-2)(y-1) + 5(y-1) = -2(y-2)$
Step 5:
Solve for y : $6(y^2 - 3y + 2) + 5y - 5 = -2y + 4 \\ 6y^2 - 18y + 12 + 7y - 9 = 0 \\ 6y^2 - 11y + 3 = 0 \\ (3y - 1)(2y - 3) = 0 \\ 3y - 1 = 0 \text{ or } 2y - 3 = 0 \\ y = \frac{1}{3}\frac{x}{2}$

YT#3 Solve (a) 2|5 - 2x| - 3 > 5

Step 1:

$$\begin{vmatrix} \text{ISOLATE absolute value: } 2|5-2x| > 8\\ |5-2x| > 4 \end{vmatrix}$$
Step 2:

$$\begin{vmatrix} \text{Think of the inside of the absolute value as a quantity Q.\\ Then talk it out in words - how far away from zero is Q?\\ Q is MORE THAN 4 units from zero, so... to the LEFT of -4 or to the RIGHT of 4. \end{vmatrix}}$$
Step 3:

$$\begin{vmatrix} \text{Make appropriate inequality statement w/o absolute value:} 5-2x < -4 & or & 5-2x > 4 \end{vmatrix}$$
Step 4:

$$\begin{vmatrix} \text{Step 3:} \\ -2x < -4 & or & 5-2x > 4 \\ (-\infty, \frac{1}{2}) \cup (\frac{9}{2}, \infty) \end{vmatrix}$$
Step 4:

$$\begin{vmatrix} \text{Step 4:} \\ -\infty, \frac{1}{2} \cup (\frac{9}{2}, \infty) \end{vmatrix}$$
Step 4:

$$\begin{vmatrix} \text{VTH4} (a) \text{ Consider the statement } |x| > 4. \\ 1) \text{ Explains what it means. Answer: x is more than 4 units of 0, or x lies beyond 5 units from 0. \\ 2) Draw a graph. \\ \hline x & -4 & 0 & -4 \\ \hline x & -4 & 0 & -4 \\ \hline x & -4 & 0 & -4 \\ \hline x & -4 & 0 & -4 \\ \hline x & -4 & 0 & -4 \\ \hline x & -4 & 0 & -4 \\ \hline x & -4 & 0 & -4 \\ \hline x & -4 & 0 & -4 \\ \hline x & -4 & 0 & -4 \\ \hline x & -4 & 0 & -x > 4 \\ \hline x &$$

(b) $\left| -\frac{2}{3}(x+4) - 2 \right| + 2 \le 6$

YT#5: Solve (a)
$$(x - 3)(2x + 1) = 3$$
 (b) $2(x - 3)^2 - 40 = 0$

Can I set each factor = 0? NO! 3? NO! Must have ZERO on right side!



What's my best move? Add 40, divide by 2, then square root both sides!

$$2(x-3)^{2} = 40$$

(x-3)^{2} = 20
x-3 = \pm\sqrt{20}
x-3 = \pm 2\sqrt{5}
x = 3 \pm 2\sqrt{5}
{3 \pm 2\sqrt{5}}

YT#7: Solve (a)
$$4a^2 - b^2 = 8$$
 for *a*.

$4a^2 = b^2 + 8$
$a^2 = \frac{b^2 + 8}{a^2 + 8}$
$\frac{4}{\sqrt{h^2 + 2}}$
$a = \pm \frac{\sqrt{b^2 + 8}}{2}$

	YT#7: Solve (b) $y = \frac{6x+1}{3x-5}$ for <i>x</i> .
Step 1:	M! both sides by LCD: $y(3x - 5) = (\frac{6x+1}{3x-5})(3x - 5)$
Step 2:	Distribute: $3xy - 5y = 6x + 1$
Step 3:	Isolate <i>x</i> terms: $3xy - 6x = 5y + 1$
Step 4:	F! out x: $x(3y-6) = 5y + 1$
Step 5:	Divide: $x = \frac{5y+1}{3y-6}$

More Equations

YT#8: Solve (a) $-4\sqrt[3]{2x-5} + 6 = 1$

Step 1: ISOLATE *u*-term:
$$-4\sqrt[3]{2x-5} = -5$$

 $\sqrt[3]{2x-5} = \frac{5}{4}$

Step 2: Raise both sides to 3^{rd} power: $2x - 5 = \frac{125}{64}$

Step 3: Subtract 15, divide by 2: $2x = \frac{125}{64} + \frac{5(64)}{64}$ $2x = \frac{125}{64} + \frac{320}{64}$ $2x = \frac{445}{64}$ $x = \frac{445}{128}$

No need to CHECK b/c we raised both sides to an ODD power!

YT#8: Solve (c) $2x^3 - 128 = 0$ (Hint: divide both sides by 2 1st!) $2x^{3} - 128 = 0$ $x^{3} - 64 = 0$ Recognize as $A^{3} - B^{3}$: $x^{3} - 64 = (x)^{3} - (4)^{3}$ $A^{3} - B^{3} = (A - B)(A^{2} + AB + B^{2})$: A = x, B = 4 $x^{3} - 64 = (x - 4)(x^{2} + 4x + 16) = 0$ $x - 4 = 0 \quad or \quad x^{2} + 4x + 16 = 0 \qquad x = \frac{-4 \pm \sqrt{16 - 48}}{2}$ $x = 4 \qquad x = \frac{-4 \pm \sqrt{-16}(2)}{2}$ $x = \frac{-4 \pm \sqrt{-16}(2)}{2}$ $x = -2 \pm 2i\sqrt{2}$

YT#8: Solve (b) $98x^3 - 49x^2 - 8x + 4 = 0$

Group terms:
$$(98x^3 - 49x^2) - (8x - 4) = 0$$

F! out GCF: $49x^2(2x - 1) - 4(2x - 1) = 0$
F! out GCF: $(2x - 1)[49x^2 - 4] = 0$
 $2x - 1 = 0 \quad or \quad 49x^2 - 4 = 0$
 $2x = 1 \quad or \quad 49x^2 = 4$
 $x = \frac{1}{2} \quad or \quad x^2 = \frac{4}{49}$
 $x = \pm \frac{2}{7}$
 $\{\frac{1}{2}, \pm \frac{2}{7}\}$

YT#9 Solve (a) $\sqrt{k-2} = \sqrt{2k+3} - 2$

(b) $(2x - 1)^{3/2} - 3 = 122$ (NO CALCULATOR!!)

Step 1:	Isolate radical: already done
Step 2:	Square both sides: $\sqrt{k-2}^2 = (\sqrt{2k+3}-2)^2$
	Square out & S! $k - 2 = 2k + 3 - 4\sqrt{2k + 3} + 4$
Step 3:	Isolate radical: $4\sqrt{2k+3} = k+9$
Step 4:	Square both sides: $\left[4\sqrt{2k+3}\right]^2 = (k+9)^2$
	$16(2k+3) = k^2 + 18k + 81$
	$32k + 48 = k^2 + 18k + 81$
	$0 = k^2 - 14k + 33$
	0 = (k - 11)(k - 3)
Step 5:	k = 11 or 3 CHECK! put in ORIGINAL EQ – BOTH work
~~~~	CHECK: put in OKIONAL EQ - DOTH WORK
	Solution: {11, 3}

Step 1: ISOLATE radical: 
$$(2x - 1)^{3/2} = 125$$
  
Step 2: Cube both sides:  $[(2x - 1)^{3/2}]^{\frac{1}{3}} = 125^{\frac{1}{3}}$   
 $(2x - 1)^{1/2} = 5$   
Step 3: Square both sides:  $2x - 1 = 25$   
 $2x = 26$   
 $x = 13$