ALGEBRA QUALIFIER AUGUST 2024

Instructions: There are 10 problems on this exam. For best results, we encourage you to attempt all 10 problems. Please use **only one side** of each sheet of paper that you turn in. Start each problem on a new page and put only your code word (not your banner ID number) on each page. Before turning the exam in, please **place the problems in order** and number all the pages. Clear and concise answers with good justification will improve your score.

(1) Suppose
$$1 \to H \xrightarrow{\alpha} G \xrightarrow{\beta} K \to 1$$
is an exact sequence of groups such that H and K are abelian and $\alpha(H)$ is central in G. Let a be a fixed element of G. Show that $\phi : G \to G$ defined by $\phi(g) = aga^{-1}g^{-1}$ is a group homomorphism.

(2) Let G be a group of order $3 \cdot 2^n$ for $n \geq 1$. Show that G is solvable.

(3) Let R be a ring satisfying $x^3 = x$ for all $x \in R$.
   (a) Prove that the characteristic of R divides 6.
   (b) Give two examples of rings, each of different characteristics which satisfy $x^3 = x$ for all $x \in R$.
   (c) If the characteristic of R is 2, show that R is Boolean, i.e. $x^2 = x$ and hence commutative.

(4) Prove that the ring $\mathbb{C}[x^3, x^4]$ is not a unique factorization domain.

(5) Let $I = (x^2, xy)$ be the ideal in the ring $\mathbb{Q}[x, y]$. Show that I is not a prime ideal. Determine two prime ideals which contain I. Are any of these ideals maximal ideals?

(6) Let $R = \mathbb{Z}/56\mathbb{Z}$. Determine all abelian groups of order at most 100 which are injective R-modules.

(7) Prove that $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}[x], \mathbb{Q})$ is uncountable.

(8) Suppose A is a 3×3 matrix with coefficients in $\mathbb{F}_7$ satisfying $A^9 = I$. Determine the **number** of conjugacy classes of A. (It is not necessary to list a matrix for each class, but think about the possible rational canonical forms such a matrix A might have.)

(9) Prove that the Galois group of the polynomial $x^4 - 3$ over $\mathbb{Q}$ is not abelian.

(10) Suppose $K \subseteq E$ are fields of characteristic 0 with $[E : K] = 72$ and K is Galois over E. Prove that there exists a nontrivial extension F of E with $E \subseteq F \subseteq K$ where F is Galois over E.