Department of Mathematics and Statistics
University of New Mexico

Real Analysis **Qualifying Exam** **August 2024**

Instructions: Please hand in all of the 7 following problems (4 in the front page and 3 in the back page). All problems have equal value. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. Clear and concise answers with good justification will improve your score.

1. Let (X, d) be a metric space. Two sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ in X are said to be equivalent if and only if for all $\epsilon > 0$ there is a natural number $N \geq 1$ such that $d(x_n, y_n) \leq \epsilon$ for all $n \geq N$. In that case we denote $\{x_n\} \sim \{y_n\}$.
   (a) Show that equivalence of sequences in a metric space is an equivalence relation.
   (b) Given $f : X \to \mathbb{R}$, show that f is uniformly continuous function on X if an only if equivalent sequences on X are mapped into equivalent sequences on $\mathbb{R}$. (Here we are viewing $\mathbb{R}$ as a metric space with the Euclidean metric.)

2. Show that if $\{a_k\}_{k=1}^{\infty}$ is a decreasing sequence of real numbers and $\sum_{k=1}^{\infty} a_k$ is convergent, then $\lim_{k\to\infty} k a_k = 0$.

3. Let f be a continuous function on $[0, 1]$ such that $f(x) > 0$ for all $x \in [0, 1]$.
   (a) Show that for every $\epsilon > 0$ there is a polynomial p such that $0 \leq p(x) \leq f(x)$ and $|p(x) - f(x)| \leq \epsilon$ for all $x \in [0, 1]$.
   (b) Show that there is a monotone increasing sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ such that $0 \leq p_n(x) \leq p_{n+1}(x)$ for all $n \geq 1$ and $p_n \to f$ uniformly on $[0, 1]$.

4. Show the following limit is zero. Justify each step of your solution.

$$\lim_{n\to\infty} \int_0^1 \frac{n\cos(x)}{1 + n^4 x^3}\, dx = 0.$$

5. Let U and V be open sets in $\mathbb{R}^n$ and let f be a one-to-one mapping from U onto V (so that there is an inverse mapping $f^{-1} : V \to U$). Suppose that f and f^{-1} are both continuous. Show that for any set S such that $\overline{S} \subset U$ and $\overline{f(S)} \subset V$ we have $f(\partial S) = \partial(f(S))$.

   (Here we are viewing $\mathbb{R}^n$ as a metric space with the Euclidean metric, and $\overline{S}$ and ∂S denote the closure and the boundary of the set S respectively.)

6. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called (positively) homogeneous of degree $s \in \mathbb{R}$ if $f(t\mathbf{x}) = t^s f(\mathbf{x})$ for all $t > 0$ and all non-zero $\mathbf{x} \in \mathbb{R}^n$. Here $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where $x_i \in \mathbb{R}$ for all $i = 1, 2 \ldots, n$. Show that if f is homogeneous of degree s, then at any point $\mathbf{x} \in \mathbb{R}^n$ where f is differentiable we have

$$x_1 \frac{\partial f}{\partial x_1}(\mathbf{x}) + x_2 \frac{\partial f}{\partial x_2}(\mathbf{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\mathbf{x}) = s\, f(\mathbf{x}).$$

7. Let $B = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}$ (the closed unit ball in $\mathbb{R}^n$). Let f and g be real-valued continuous functions on B with $g \geq 0$. Then there is a point $\mathbf{a} \in B$ such that

$$\int_B f(\mathbf{x})\, g(\mathbf{x})\, d^n\mathbf{x} = f(\mathbf{a}) \int_B g(\mathbf{x})\, d^n\mathbf{x}.$$

   Here $d^n\mathbf{x} = dx_1 \ldots dx_n$, and the integral is a Riemann integral.