Complex Analysis Qualifying Exam
August 9, 2024

Instructions: Hand in all of the following 8 problems. Start each problem on a new page, number the pages, and put only your code word (not your banner ID number) on each page. **Submit your solutions arranged in numerical order**. Clear and concise answers with good justification will improve your score.

1. Let Ω be open and connected in $\mathbb{C}$ and let $f, f_1, f_2, \ldots, f_n$ be holomorphic in Ω.
   (a) Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$
   (b) Show that
   $$|f_1(z)|^2 + |f_2(z)|^2 + \cdots + |f_n(z)|^2$$
   is harmonic in Ω if and only if each f_k is constant.

2. Consider the function
   $$F(z) = \oint_{|\zeta|=2} \frac{d\zeta}{\zeta(\zeta - z)(\zeta - z + 1)}.$$
   Determine the limit of $F(z)$ as $z \to 2$ from inside the circle $|\zeta| = 2$ and also from outside the circle $|\zeta| = 2$. Is $F(z)$ continuous at $z = 2$?

3. Let a be a nonzero real number. Evaluate
   $$\int_0^\infty \frac{\cos(ax)}{1 + x^2}\, dx.$$
   Explain all steps carefully (show contours, etc.).

4. Let k be a positive integer. Characterize all the holomorphic functions on $\mathbb{C} \setminus \{0\}$ satisfying $|f(z)| \le |z|^{-k}$ for all $z \neq 0$. Prove your answer.

5. Suppose f is holomorphic and bounded on $\{z : |z| > R\}$ for some $R > 0$. Prove that the Laurent series of f centered at 0 has the form
   $$f(z) = \sum_{n=-\infty}^{0} a_n z^n,$$
   that is, all the coefficients of z^n in the Laurent series with $n > 0$ vanish.

6. Let f be holomorphic and bounded in the disc $D(0, R)$ for some $R > 0$ so that $M = \sup_{|z|<R} |f(z)|$ is finite. Suppose that for some positive integer k
   $$f(0) = f'(0) = \cdots = f^{(k)}(0) = 0.$$
   Prove that
   $$|f(z)| \le \frac{M}{R^{k+1}} |z|^{k+1}.$$

7. Find the number of solutions to $2z^5 + 8z - 1 = 0$ lying in the annulus $\{z : 1 < |z| < 2\}$. Justify your answer.

8. Let $U = \{z \in \mathbb{C} : |z| < 2, |z-1| > 1\}$ and $V = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$.

   (a) Find a conformal map from U to V.

   (b) Find a solution $u(x, y)$ to the Laplace equation $\Delta u = 0$ on U with $u = 3$ on the inner circle $|z-1| = 1$ and $u = 1$ on the outer circle $|z| = 2$.