Fall 2024 Numerical Analysis MS/PhD Qualifying Examination

Please write your code number (not your name) on each work sheet. Please do five of the following six problems, providing concise answers with justification. Please mark an X through the problem you do not want graded.

1. (20 points)

(a) List the the properties defining a norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$.

(b) Prove the Cauchy-Schwarz inequality for $\mathbb{R}^n$. That is, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then show

$$|\mathbf{x}^T\mathbf{y}| \leq \sqrt{(\mathbf{x}^T\mathbf{x})(\mathbf{y}^T\mathbf{y})}.$$

(c) The "p-norm" $\|\cdot\|_p$ on $\mathbb{R}^n$ is defined as

$$\|\mathbf{x}\|_p = \Big(\sum_{k=1}^{n} |x_k|^p\Big)^{1/p}.$$

Show that the 2-norm (i.e. $p = 2$) satisfies the properties of a norm.

(d) Show (e.g. by giving a counterexample) that the 0.5-norm (i.e. $p = \frac{1}{2}$) does not satisfy all properties of a norm on $\mathbb{R}^2$. Is there an n for which the 0.5-norm satisfies the properties of a norm? Justify your answer.

2. (20 points) Let $\|\cdot\|$ be the infinity norm, with $\kappa(\cdot)$ the associated condition number, and consider $A\mathbf{x} = \mathbf{b} \neq \mathbf{0}$ for $A \in \mathbb{R}^{n\times n}$ invertible.

(a) Consider the solution $\hat{\mathbf{x}}$ to the perturbed linear system $(A+\delta A)\hat{\mathbf{x}} = \mathbf{b}$, where $\delta A \in \mathbb{R}^{n\times n}$ obeys $\|A^{-1}\|\|\delta A\| < 1$. Show that

$$\mathbf{x} - \hat{\mathbf{x}} = (I + A^{-1}\,\delta A)^{-1}A^{-1}\,\delta A\mathbf{x},$$

and that the relative forward error $\|\mathbf{x} - \hat{\mathbf{x}}\|/\|\mathbf{x}\|$ satisfies

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq \frac{\kappa(A)}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}}\Big(\frac{\|\delta A\|}{\|A\|}\Big).$$

Suppose that $\|\delta A\|$ is small relative to $\|A\|$. Is the relative forward error also small? You may use the result that if a square matrix M obeys $\|M\| < 1$, then $I - M$ is invertible and

$$\|(I - M)^{-1}\| \leq \frac{1}{1 - \|M\|}.$$

(b) Consider solving the system $A\mathbf{x} = \mathbf{b}$, assuming that that the LU-factorization $A = LU$ computed by Gaussian elimination with partial pivoting (GEPP) features no permutation matrix. Alternatively, if GEPP produces the factorization $PA = LU$, you may redefine A,$\mathbf{b}$ as the permuted quantities PA,$P\mathbf{b}$. Denote the numerically computed factors as $\hat{L}$ and $\hat{U}$, where $\hat{L}$ is unit lower triangular (as is L), and $\hat{U}$ is upper triangular (as is U). Note that, due to rounding errors, $\hat{L}\hat{U} \neq A$. Assuming that $n \ll 1/\epsilon$, where ϵ is the machine precision (or unit roundoff), the numerical solution $\hat{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$ obtained from the factorization obeys

$$(A + E)\hat{\mathbf{x}} = \mathbf{b}, \qquad \|E\| = O(\epsilon)\|\hat{L}\|\|\hat{U}\|.$$

Show that $\|\hat{L}\| = O(n)$, in effect $O(1)$ with the dimension n viewed as fixed. What bound on $\|\hat{U}\|$ then ensures that the estimate from **(a)** implies $\|\mathbf{x} - \hat{\mathbf{x}}\|/\|\mathbf{x}\| = O(\kappa(A)\epsilon)$? Be specific if possible. Will the relative foward error then be necessarily small?

3. (20 points) A real Householder transformation $H \in \mathbb{R}^{n\times n}$ is a symmetric orthogonal matrix of the form $H = I - 2\mathbf{u}\mathbf{u}^T$, where $\mathbf{u} \in \mathbb{R}^n$.

(a) What additional property does $\mathbf{u}$ satisfy? Show that H is orthogonal.

(b) Given $\mathbf{x} \in \mathbb{R}^n$, consider a Householder transformation H such that $H\mathbf{x} = (c, 0, 0, \ldots, 0)^T$. How is c related to $\|\mathbf{x}\|_2$? Moreover, given that $H = I - 2\mathbf{u}\mathbf{u}^T$, derive a formula for the vector $\mathbf{u}$ such that $H\mathbf{x} = (c, 0, 0, \ldots, 0)^T$.

(c) What is a QR-decomposition of a matrix $A \in \mathbb{R}^{m\times n}$? Be precise in your description.

(d) Describe how you can use Householder transformations to compute a QR-decomposition of a full-rank matrix $A \in \mathbb{R}^{m\times n}$, $m \geq n$. State the computational cost in big-O notation.

(e) How can QR-decomposition be used to solve the linear least-squares problem $A\mathbf{x} = \mathbf{b}$ efficiently, where $A \in \mathbb{R}^{m\times n}$ is full-rank and $m \geq n$? If already given a QR-decomposition, what is the additional computational cost of solving the linear least-squares problem?

4. (20 points) Let $A \in \mathbb{R}^{n\times n}$ be a *singular matrix*, and consider the ill-posed problem $A\mathbf{x} = \mathbf{b}$. *Tikhonov regularization* trades this problem for the system $(A^TA + \mu^2 I)\mathbf{x} = A^T\mathbf{b}$, where $\mu \neq 0$ is a chosen parameter.

(a) Prove that the coefficient matrix $A^TA + \mu^2 I$ for the regularized problem is nonsingular. Be sure to support any assertions.

(b) How can Tikhonov regularization be implemented without using A^T? *Hint: consider the SVD of A.*

5. (20 points) Let $A \in \mathbb{R}^{n\times n}$ have entries a_{ij} which satisfy

$$a_{ii} \geq \sum_{j\neq i} |a_{ij}| + 2, \qquad 5 \leq a_{ii} \leq 7.$$

(a) Prove that A^{-1} exists.

(b) Find a numerical bound for $\|A\|_\infty$.

(c) Now assume $A = A^T$. Find numerical bounds on $\|A\|_2$ and $\|A^{-1}\|_2$.

6. (10 points) Let $A \in \mathbb{R}^{n\times n}$ be symmetric postive definite, and consider the system $A\mathbf{x} = \mathbf{b}$. The kth Krylov subspace is $\mathcal{K}_k(A, \mathbf{b}) = \text{span}\{\mathbf{b}, A\mathbf{b}, A^2\mathbf{b}, \ldots, A^{k-1}\mathbf{b}\}$ for $k \leq n$. You may assume $\dim\mathcal{K}_k(A, \mathbf{b}) = k$ for $1 \leq k \leq n$. Suppose $Q_k \in \mathbb{R}^{n\times k}$ has orthonomal columns which span $\mathcal{K}_k(A, \mathbf{b})$, recursively defined such that $Q_k = [Q_{k-1}, \mathbf{q}_k]$, $Q_1 = \mathbf{b}/\|\mathbf{b}\|_2$.

(a) Show that $H_k = Q_k^T A Q_k \in \mathbb{R}^{k\times k}$ is nonsingular.

(b) Given an approximate solution $\mathbf{x}_k \in \mathcal{K}_k(A, \mathbf{b})$ and its residual $\mathbf{r}_k = \mathbf{b} - A\mathbf{x}_k$, the *Galerkin condition* is $\mathbf{z}^T\mathbf{r}_k = 0$, $\forall \mathbf{z} \in \mathcal{K}_k(A, \mathbf{b})$. Prove that the Galerkin condition is equivalent to $\mathbf{x}_k = \|\mathbf{b}\|_2 Q_k H_k^{-1}\mathbf{e}_1$, where $\mathbf{e}_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^k$.

(c) Prove that, over the kth Krylov subspace, $\mathbf{x}_k$ from **(b)** minimizes the A^{-1}–norm of the residual, that is, it minimizes $\|\mathbf{r}\|_{A^{-1}} \equiv \sqrt{\mathbf{r}^T A^{-1}\mathbf{r}}$, where $\mathbf{r} = \mathbf{b} - A\mathbf{z}$ for $\mathbf{z} \in \mathcal{K}_k(A, \mathbf{b})$.